Section 6.3
A Planarity Algorithm

PIECES OF A GRAPH

• If $G_1 = (V_1, E_1)$ is a subgraph of $G = (V, E)$, then a piece of $G$ relative to $G_1$ is either an edge $e = uv$ where $e \in E_1$ and $u, v \in V(G_1)$ or a connected component of $(G - G_1)$ plus any edges incident to vertices of this component.

• For any piece $P$ relative to $G_1$, the vertices of $P$ in $G_1$ are called contact vertices.

• If a piece has two or more contact vertices, it is called a segment.

• Two segments are incompatible if when placed in the same region of the plane determined by a cycle $C$, at least two of their edges cross. Note that when embedded $C$ divides the plane into two regions, one interior to $C$, the other exterior.

PRELIMINARY TEST TO SIMPLIFY FINDING A PLANAR EMBEDDING

1. If $|E| > 3p - 6$, then the graph must be nonplanar.

2. If the graph is disconnected, consider each component separately.

3. If the graph contains a cut vertex, then it is clearly planar if and only if each of the blocks is planar. Thus, we can limit our attention to 2-connected graphs.

4. Loops and multiple edges change nothing; hence, we need only consider graphs.

5. A vertex of degree 2 can certainly be replaced by an edge joining its neighbors. This contraction of all vertices of degree 2 constructs a homeomorphic graph with the smallest number of vertices. Certainly, a graph is planar if and only if the contraction is planar.
**G-ADMISSIBLE**

Let $\hat{H}$ be a plane embedding of a subgraph $H$ of $G$. If there exists a plane embedding of $G$ (say $\hat{G}$) such that $\hat{H} \subseteq \hat{G}$, then $\hat{H}$ is said to be $G$-admissible.

**SEGMENTS AND SUBGRAPHS**

- Let $S$ be any segment of $G$ relative to a subgraph $H$. $S$ can be drawn in region $r$ of $\hat{H}$ provided all the contact vertices of $S$ lie in the boundary of $r$.
- We can extend the embedding of $\hat{H}$ to include at least part of $S$.

**STRATEGY OF THE DEMOUCRON, MALGRANGE, AND PERTUISET ALG.**

- Find a sequence of subgraphs $\hat{H}_1, \hat{H}_2, ..., \hat{H}_{|E|-p+2} = G$ such that $H_i \subset H_{i+1}$ and such that $\hat{H}_i$ is $G$-admissible (if possible).
- We either construct a plane embedding of $G$ (if one is possible) or discover some segment $S$ which cannot be compatibly embedded in any region.
SOME NOTATION

Given a plane embedded subgraph \( \tilde{H}_i \), for each segment \( S \), the set \( R(S, \tilde{H}_i) \) is defined to be the set regions in which \( S \) can be compatibly embedded in \( \tilde{H}_i \).

DMP PLANARITY ALGORITHM

Algorithm 6.3.1 DMP Planarity Algorithm

Input: A preprocessed block (after applying tests 1-5).

Output: The fact that the graph is planar or nonplanar.

Method: Look for a sequence of admissible embeddings beginning with some cycle \( C \).

DMP ALGORITHM (CONCLUDED)

1. Find a cycle \( C \) and a planar embedding of \( C \) as the first subgraph \( \tilde{H}_1 \).
   Set \( i \leftarrow 1 \) and \( r \leftarrow 2 \).
2. If \( r = \lvert E \rvert - p + 2 \), then stop;
   else determine all segments \( S \) of \( \tilde{H}_i \) in \( G \) and for each segment \( S \) determine \( R(S, \tilde{H}_i) \).
3. If there exists a segment \( S \) with \( R(S, \tilde{H}_i) = \emptyset \),
   then stop and say \( G \) is nonplanar;
   else if there exists a segment \( S \) such that \( \lvert R(S, \tilde{H}_i) \rvert = 1 \),
   then let \( R = R(S, \tilde{H}_i) \);
   else let \( S \) be any segment and \( R \) be any region in \( R(S, \tilde{H}_i) \).
4. Choose a path \( P \) in \( S \) connecting two contact vertices. Set \( H_{i+1} = H_i \cup P \) to obtain the embedding \( \tilde{H}_{i+1} \) with \( P \) placed in \( R \).
5. Set \( i \leftarrow i + 1 \), \( r \leftarrow r + 1 \) and go to step 2.