Section 6.2
Characterizations of Planar Graphs

**EDGE SUBDIVISION AND HOMEOMORPHIC**
- By a *subdivision* of an edge $e = xy$, we mean that the edge $xy$ is removed from the graph and a new vertex $w$ is inserted in the graph along with the edges $wx$ and $wy$.
- A graph $H$ is *homeomorphic from* $G$ if either $H$ is isomorphic to $G$ or $H$ is isomorphic to a graph obtained by subdividing some sequence of edges of $G$.
- A graph $G$ is *homeomorphic with* $H$ if both $G$ and $H$ are homeomorphic from a graph $F$.
- “Homeomorphic with” is an equivalence relation.

**SOME COMMENTS**
- If a graph is planar, any graph obtained by subdividing edges is planar since all the added vertices have degree 2.
- If a graph is planar, then the graph obtained by *contracting* the vertices of degree 2 (replacing every vertex of degree 2 by an edge between its two neighbors) is also planar.
- Thus, a graph is planar if and only if all graphs homeomorphic with it are planar.
Currently, which graphs do we know are nonplanar?

- \( K_5 \)
- \( K_{3,3} \)
- Graphs containing \( K_5 \) or \( K_{3,3} \) as a subgraph.
- Graphs containing a subgraph homeomorphic with \( K_5 \) or \( K_{3,3} \).

Kuratowski showed that up to homeomorphic graphs \( K_5 \) or \( K_{3,3} \) are the only subgraphs that cause a graph to be nonplanar!!!!!
OUTLINE OF PROOF (CONTINUED)

(⇐) Suppose $G$ is a graph that contains no subdivision of $K_{3,3}$ or $K_5$. Here are the steps used to prove the result.

1. Prove that $G$ is planar if and only if each block of $G$ is planar.

2. Explain why it suffices to show that a block is planar if and only if it contains no subdivision of $K_{3,3}$ or $K_5$. Assume $G$ is itself a nonplanar block of minimum size (connected with no cut vertex).

OUTLINE OF PROOF (CONTINUED)

3. Suppose that $G$ is a nonplanar block that contains no subdivision of $K_{3,3}$ or $K_5$ (and search for contradiction).

4. Prove $\delta(G) \geq 3$.

5. Establish the existence of an edge $e = uv$ such that the graph $G - e$ is also a block.

6. Explain why $G - e$ is a planar graph containing a cycle $C$ that includes both $u$ and $v$, and choose $C$ to have the maximum number of interior regions.

OUTLINE OF PROOF (CONCLUDED)

7. Establish several structural facts about the subgraphs inside and outside the cycle $C$.

8. Use these structural facts to demonstrate the existence of a subdivision of $K_{3,3}$ or $K_5$, thus establishing the contradiction (from step 3).