Section 3.1
Fundamental Properties of Trees

TREES
Recall, that a tree is a connec acyclic graph.

TWO CHARACTERIZATIONS OF TREES

Theorem 3.1.1: A graph $T$ is a tree if and only if every two distinct vertices of $T$ are joined by a unique path.

Theorem 3.1.2: A $(p, q)$ graph $T$ is a tree if and only if $T$ is connected and $q = p - 1$. 
**CHARACTERIZATIONS OF TREES**

**Theorem 3.1.3:** The following are equivalent on a \((p,q)\) graph \(T\):

1. The graph \(T\) is a tree.
2. The graph \(T\) is connected and \(q = p - 1\).
3. Every pair of distinct vertices of \(T\) is joined by a unique path.
4. The graph \(T\) is acyclic and \(q = p - 1\).

**VERTICES AND EDGES IN TREES**

- In any tree of order \(p \geq 3\), any vertex of degree at least 2 is a cut vertex. There are at least two vertices of degree 1.
- The vertices of degree 1 in a tree are called **end vertices** or **leaves**. The remaining vertices are called **internal vertices**.
- Every edge in a tree is a bridge.

**SPANNING TREES**

- Every connected graph \(G\) contains a spanning subgraph that is a tree, called a **spanning tree**.
- A spanning tree that preserves the distance from \(v\) to each of the remaining vertices in \(G\) is said to be **distance preserving from \(v\)** or **\(v\)-distance preserving**.
DISTANCE PRESERVING TREES

**Theorem 3.1.4:** For every vertex $v$ of a connected graph $G$, there exists a $v$-distance preserving tree.

MANY TREES EMBEDDED AS SUBGRAPHS

**Theorem 3.1.5:** Let $G$ be a graph with $\delta(G) \geq m$ and let $T$ be any tree of order $m + 1$. Then $T$ is a subgraph of $G$. 