**Section 11.1: Sequences**

**Introduction:**

* A ***sequence*** is a function whose domain is the set of positive integers.
* The functions values are called the ***terms*** of the sequence.
* The ***value*** of function at the integer is .
* The variable is called the ***index***.

A sequence may be specified in three ways:

* By an explicit formula

* By a recursive formula

* By giving enough terms to establish a pattern

 6, 18, 54, 162, . . .

**Graphing a Sequence on the TI-83/84:**

 1. Press **MODE**. Select **Seq** at the end of the fourth line.

 2. Press **Y=** and type the sequence. Use the ***X,T,θ,n*** key to get the variable “*n*.”

 3. Adjust the viewing window as necessary.

Example: Graph the sequence .

**The Limit of a Sequence:**

 Consider the sequence .



 It appears that

NOTE: All the limit theorems for functions (learned in Calculus I) also apply to sequences.

**Definition:** If is finite, then the sequence ***converges***; if is infinite or does not exist, then the sequence ***diverges***.

 Observe the if is a positive number, then

Examples: Show the following sequences converge.

1.

2.

**Theorem:** Let be a sequence and let *f* be a function defined on such that for *n*.

If , then .

However, it is NOT necessarily true that if , then . For example:

 If , then does not exist. See the graph below.



The graph of *y* = sin *πx*

If , then . This is because is a succession of zeros for

 . See the graph below.



The graph of (in red)

with (in blue) superimposed.

Example: Does converge?

**Some Convergence Theorems:**

**The Squeeze Theorem:** Suppose the and both converge to *L* and that for (*K* is a fixed positive integer). Then converges to *L*.

Example: Use the Squeeze Theorem to show that converges.

The Squeeze Theorem can be used to prove the Absolute Value Theorem stated below.

**Absolute Value Theorem:** If , then .

Example: Show that if , then the sequence converges.

Solution:

 If , . So, we only need to deal with cases when and ; that is, when .

 Since , . Thus, there is some positive number such that .

 Now, we recall the Binomial Formula which says

Note that: .

Using the Binomial Formula, we see that

Hence, we know that

and that

Thus, by the Squeeze Theorem, . And, by the Absolute Value Theorem, .

**Monotonic Sequences:**

* A sequence is called ***nondecreasing*** if .
* A sequence is called ***nonincreasing*** if .
* A sequence is called ***monotonic*** if it is either nondecreasing or nonincreasing.

Two examples of nondecreasing sequences are

A nondecreasing sequence can do one of two things:

 1. March off to infinity, or

 2. If it is bounded above (that is, for and some fixed number *K*), then it must bump against a “lid.” See the figure below.



A nondecreasing sequence that

is bounded above by 2.

NOTE: Sequence above marches off to infinity. However, sequence above is bounded above by 1 and has limit 1.

**Bounded Sequences:**

1. A sequence is ***bounded above*** if there is a real number *M* such that for all . The number *M* is called an ***upper bound*** of the sequence.

2. A sequence is ***bounded below*** if there is a real number *N* such that for all . The number *N* is called a ***lower bound*** of the sequence.

3. A sequence is ***bounded*** if it is bounded above and bounded below.

**Monotonic Sequence Theorem:** Every bounded, monotonic sequence is convergent.

NOTE: In the theorem above it is not necessary that the sequence be monotonic initially, only that they be monotonic from some point on—that is, for . In fact, *the convergence or divergence of a sequence does not depend on the character of its initial terms but rather on what is true for large n*.

Example: Use the Monotonic Sequence Theorem to show that

converges.