Section 6.3: 3/28/05

Solutions About Ordinary Points

Ex 1

$$y'' + (e^x)y' + (\sin x)y = 0$$

$x=0$ is an ordinary point since

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ converg for all finite values of x.

Ex 2 $\quad x y'' + (\sin x) y = 0$

$$y'' + \frac{\sin x}{x} y = 0 \qquad ; \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

Ex 3 $\quad y'' + \ln|x| y = 0$

$Q(x) = \ln|x|$ is not analytic at $x=0$.

$\therefore x=0$ is a singular point.

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In general $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

$a_2(x), a_1(x), a_0(x)$ with no common factors, a point $x = x_0$ is

if $a_2(x_0) \neq 0 \Rightarrow x_0$ is an ordinary point.

$a_2(x_0) = 0 \Rightarrow x_0$ is a singular point.

Ex 4 $(x^2-1)y'' + 2xy' + 6y = 0$

(a) Singular point @ $x^2 - 1 = 0$

$x = \pm 1$

All other finite values of x are ordinary points.

(b) $(x^2+1)y'' + xy' - y = 0$

Singular points when $x^2+1 = 0$

$x = \pm i$

All other finite values of x, real or complex, are ordinary points.

Ex 5 Cauchy Euler Equation

$ax^2y'' + bxy' + cy = 0$

$y'' + \frac{b}{a} \cdot \frac{1}{x} y' + \frac{c}{x^2} = 0$

Singular point @ $x = 0$

All other finite values of x, real or complex, are ordinary points.

Ex 6 3/28/05

Solve $y'' - 2xy = 0$

$x=0$ is an ordinary point.

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\Rightarrow y'' - 2xy = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} 2 c_n x^{n+1}$$

$$= 2c_2 + \underbrace{\sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} 2 c_n x^{n+1}}_{\text{both series start with } x}$$

$$= 2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} 2 c_{n-1} x^n$$

$$= 2c_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) c_{n+2} - 2c_{n-1} \right] x^n = 0$$

$$= 2c_2 + \sum_{n=1}^{\infty} \left((n+2)(n+1) c_{n+2} - 2c_{n-1} \right) x^n = 0, \quad x^n \neq 0$$

$$\Rightarrow 2c_2 = 0 \Rightarrow \boxed{c_2 = 0}$$

$$(n+2)(n+1) c_{n+2} - 2c_{n-1} = 0$$

$$c_{n+2} = \frac{2c_{n-1}}{(n+2)(n+1)}$$

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$$C_{n+2} = 2\frac{C_{n-1}}{(n+2)(n+1)}$$

$$n=1, \quad C_3 = 2\frac{C_0}{3\times 2} = \frac{C_0}{3}$$

$$n=2, \quad C_4 = 2\frac{C_1}{4\times 3} = \frac{C_1}{6}$$

$$n=3, \quad C_5 = 2\frac{C_2}{5\times 4} = \frac{C_2}{10} = 0; \quad C_2 = 0$$

$$n=4, \quad C_6 = 2\frac{C_3}{6\times 5} = \frac{1}{15}C_3 = \frac{1}{45}C_0$$

$$n=5, \quad C_7 = 2\frac{C_4}{7\times 6} = \frac{1}{21}C_4 = \frac{1}{21\times 6}C_1 = \frac{1}{126}C_1$$

$$n=6, \quad C_8 = 2\frac{C_5}{8\times 7} = \frac{1}{28}C_5 = \frac{1}{28}C_5 = 0.$$

$$n=7, \quad C_9 = 2\frac{C_6}{9\times 8} = \frac{1}{36}\times\frac{C_0}{45} = \frac{1}{36}\cdot\frac{1}{45}C_0$$

$$n=8, \quad C_{10} = 2\frac{C_7}{10\times 9} = \frac{1}{45}\frac{C_1}{126}$$

$$n=9, \quad C_{11} = 2\frac{C_8}{11\times 10} = 0$$

$$n=10 \quad C_{12} = 2\frac{C_9}{12\times 11} = \frac{1}{6\times 11}\times\frac{1}{36}\cdot\frac{1}{45}C_0$$

$$= \frac{1}{66}\times\frac{1}{36}\times\frac{1}{45}C_0$$

$$Y = C_0 + C_1x + 0 + \frac{C_0}{3}x^3 + \frac{C_1}{6}x^4 + 0 + \frac{1}{45}C_0x^6 + \frac{1}{126}C_1x^7$$

$$+0 \quad \frac{1}{36}\cdot\frac{1}{45}C_0x^9 + \frac{1}{45}\times\frac{1}{126}C_1x^{10} + 0 + \frac{1}{66}\times\frac{1}{36}\times\frac{1}{45}C_0x^{11}$$

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$$Y = C_0\left[1+\frac{x^3}{3}+\frac{1}{45}x^6+\frac{1}{36}\cdot\frac{1}{45}x^9+\frac{1}{66}\cdot\frac{1}{36}\cdot\frac{1}{45}x^{11}+\cdots\right]$$

$$+C_1\left[x+\frac{x^4}{6}+\frac{1}{126}x^7+\frac{1}{45}\cdot\frac{1}{126}x^{10}+\cdots\right]$$

$$Y = Y_1 + Y_2$$

$$Y_1 = C_0\left[1+\sum_{k=1}^{\infty} 2^k\frac{(1\cdot 4\cdot 7 \cdots (3k-2))\,x^{3k}}{(3k)!}\right]$$

$$+C_1\left[x+\sum_{k=1}^{\infty} 2^k\frac{(2\cdot 5\cdot 8 \cdots (3k-1))\,x^{3k+1}}{(3k+1)!}\right]$$

Ex 7 Discovering a Polynomial Solution 3/28/05

Solve $(x^2+1)y'' + xy' - y = 0$ —(1)

$$y'' + \frac{x}{x^2+1}y' - \frac{1}{x^2+1}y = 0$$

Singular Points are $x = \pm i$

a power Series solution exist for $|x|<1$, $\text{Mod}|\pm i| = 1 = R$

$|x-0| < R$

Assume $Y = \sum_{n=0}^{\infty} C_n X^n$

$$Y' = \sum_{n=1}^{\infty} n C_n X^{n-1} \quad , \quad Y'' = \sum_{n=2}^{\infty} n(n-1) C_n X^{n-2}$$

Substitute into (1)

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1) C_n X^{n-2} + X\sum_{n=1}^{\infty} n C_n X^{n-1} - \sum_{n=0}^{\infty} C_n X^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) C_n X^n + \sum_{n=2}^{\infty} n(n-1) C_n X^{n-2} + \sum_{n=1}^{\infty} n C_n X^n - \sum_{n=0}^{\infty} C_n X^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n X^n + \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} X^n + \sum_{n=1}^{\infty} n C_n X^n - \sum_{n=0}^{\infty} C_n X^n = 0$$

$$\left\{ \begin{aligned} & 2C_2 + 6C_3 X + C_1 X - C_0 - C_1 X \\ & + \sum_{n=2}^{\infty} \left[n(n-1) C_n + (n+2)(n+1) C_{n+2} + n C_n - C_n \right] X^n \end{aligned} \right\} = 0 , \forall x$$

$$\Rightarrow \quad 2C_2 - C_0 = 0 \Longrightarrow C_2 = \frac{C_0}{2}$$

$$6C_3 = 0 \Longrightarrow C_3 = 0$$

$$C_n \left[n(n-1) + (n-1) \right] + (n+2)(n+1) C_{n+2} = 0$$

$$C_n (n-1)(n+1) + (n+2)(n+1) C_{n+2} = 0$$

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$$\Longrightarrow C_{n+2} = -\frac{(n-1)}{(n+2)} C_n \quad ; n \neq 1$$

$$C_{n+2} = \frac{(1-n)C_n}{(n+2)} \quad , n = 2, 3, \ldots$$

$n=2$, $\quad C_4 = \frac{-C_2}{4} = \frac{-C_0}{8} = -\frac{1}{2^2}\frac{1}{2!}C_0$

$n=3$, $\quad C_5 = -\frac{2}{5}C_3 = 0$

$n=4$, $\quad C_6 = -\frac{3}{6}C_4 = \frac{3}{6} \times \frac{1}{2^2}\frac{1}{2!}C_0$

$$C_6 = \frac{1}{2^3} \times \frac{3}{3!}C_0$$

$n=5$, $\quad C_7 = -\frac{4}{7}C_5 = 0$

$n=6$, $\quad C_8 = -\frac{5}{8}C_6 = -\frac{5}{8} \times \frac{1}{2^3} \times \frac{3}{3!}C_0$

$$= -\frac{1.3.5}{2^4 . 4!}C_0$$

$n=7$ $\quad C_9 = 0$

$n=8$ $\quad C_{10} = +\frac{1.3.5.7}{2^5\ 5!}C_0$

$$\therefore \quad Y = \sum_{n=0}^{\infty} C_n X^n = C_0 + C_1X + C_2X^2 + C_3X^3 + C_4X^4 \ldots$$

$$= C_0 + \frac{C_0}{2}X^2 - \frac{1}{2^2\,2!}C_0X^4 + \frac{1.3}{2^3\,3!}C_0X^6 - \frac{1.3.5}{2^4.4!}C_0X^8$$

$$+ \frac{1.3.5.7}{2^5.5!}C_0X^{10} \ldots$$

$$Y = C_1X + C_0\left[\frac{X^2}{2} - \frac{1}{2^2 2!}X^4 + \frac{1.3}{2^3 3!}X^6 - \frac{1.3.5}{2^4 4!}X^8 + \frac{1.3.5.7}{2^5 5!}X^{10} - \right.$$

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cancel{c_3 x^3} + c_4 x^4 + \cdots$$ 3/22/03

$$= c_1 x + c_0\left[1 + \frac{x^2}{2}\right] + c_4 x^4 + c_6 x^6 + c_8 x^8 + \cdots$$

$$= c_1 x + c_0\left(1 + \frac{x^2}{2}\right) + c_0 \sum_{n=2}^{\infty}\left[\frac{(-1)^{n-1} x^{2n} (1,3,5,7 \cdots (2n-3))}{2^n n!}\right], |x| < 1$$

∴ The two independent solution are

$$y_1 = c_1 x$$

$$y_2 = c_0\left(1 + \frac{x^2}{2}\right) + c_0 \sum_{n=2}^{\infty} (-1)^{n-1} \frac{x^{2n}(1,3,5 \cdots (2n-3))}{2^n n!}$$

$; |x| < 1$

Example 8: A Three Term Recurrence Relation

Solve $y'' - (1+x)y = 0$ ——— (1)

$x = 0$ is an ordinary point.

Assume a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substitute in (1)

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - (1+x) \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

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$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=0}^{\infty} C_n x^n - \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n - \sum_{n=0}^{\infty} C_n x^n - \sum_{n=1}^{\infty} C_{n-1} x^n = 0$$

$$2 \cdot 1\, C_2 - C_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) C_{n+2} - C_n - C_{n-1} \right] x^n = 0 \quad \forall x$$

$$2C_2 - C_0 = 0 \implies C_2 = \frac{C_0}{2} = \frac{C_0}{2!} \quad \text{—(2)}$$

$$(n+2)(n+1) C_{n+2} - C_n - C_{n-1} = 0$$

$$C_{n+2} = \frac{C_n + C_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3 \ldots$$

$n=1$, $C_3 = \dfrac{C_1 + C_0}{3 \times 2}$ To simplify the iteration choose $C_0 \neq 0$, $C_1 = 0$.

$$C_3 = \frac{C_0}{6} = \frac{C_0}{3!}$$

$n=2$, $C_4 = \dfrac{C_2 + C_1}{4 \times 3} = \dfrac{C_2}{12} = \dfrac{C_0}{24} = \dfrac{C_0}{4!}$

$n=3$, $C_5 = \dfrac{C_3 + C_2}{5 \times 4} = \left(\dfrac{C_0}{6} + \dfrac{C_0}{2} \right) \cdot \dfrac{1}{20}$

$$= \frac{2C_0}{3} \cdot \frac{1}{20} = \frac{C_0}{30}$$

$$\wedge\; Y_1(x) = C_0 \left[1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^5}{30} + \ldots \right]$$

to get $Y_2(x)$ Now assume $C_0 = 0$, $C_1 \neq 0$

$(2) \implies C_2 = 0$

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The recurrence formula was:

$$C_{n+2} = \frac{C_n + C_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

Assume $C_0 = 0$, $C_1 \neq 0$

$(2) \Longrightarrow C_2 = 0.$

$n=1, \quad C_3 = \frac{C_1 + C_0}{3 \times 2} = \frac{C_1}{6}$

$n=2, \quad C_4 = \frac{C_2 + C_1}{4 \times 3} = \frac{C_1}{12}$

$n=3, \quad C_5 = \frac{C_3 + C_2}{5 \times 4} = \frac{C_3}{20} = \frac{C_1}{120}$

And so on.

$\longrightarrow Y_2 = C_1 \left[x + \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{1}{120} x^5 + \cdots \right]$

$Y_{total} = Y_1 + Y_2$

$= C_0 \left[1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^5}{30} + \cdots \right]$

$+ C_1 \left[x + \frac{x^3}{6} + \frac{1}{12} x^4 + \frac{1}{120} x^5 + \cdots \right]$

Comment on How To Evaluate $C_0 + C_1$

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Ex 9 DE with a Nonpolynomial Coefficient

Solve $y'' + (\cos x)y = 0$ $x=0$ is a ordinary pt

Recall that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\cos x = \sum_{n=0} (-1)^n \frac{x^{2n}}{2n!}$$; $x=0$ is an ordinary pt.

So let us assume that the solution $y = \sum_{n=0}^{\infty} c_n x^n$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} \sum_{n=0}^{\infty} c_n x^n = 0 \quad \forall x.$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right) = 0$$

$$\left(2c_2 + 6c_3 x + 20 c_5 x^3 + \cdots + 12 c_4 x^2\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right) = 0$$

$$(2c_2 + c_0) + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{c_0}{2}\right)x^2 + \left(20c_5 + c_3 - \frac{c_1}{2}\right)x^3 + \cdots = 0 \quad \forall x$$

$$\left.\begin{array}{l} 2c_2 + c_0 = 0 \Rightarrow c_2 = -\frac{c_0}{2} \\ 6c_3 + c_1 = 0 \Rightarrow c_3 = -\frac{c_1}{6} \end{array}\right\} c_0, c_1 \text{ are arbitrary}$$

$$12c_4 + c_2 - \frac{c_0}{2} = 0 \Rightarrow c_4 = \frac{c_0}{12}$$

$$20c_5 + c_3 - \frac{c_1}{2} = 0 \Rightarrow 20c_5 = \frac{2c_1}{3} \Rightarrow c_5 = \frac{c_1}{30}$$

$$\therefore y = c_0\left[1 - \frac{x^2}{2} + \frac{1}{12}x^4 - \cdots\right] + c_1\left[x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots\right]$$

No Singular Point $\Rightarrow$ Series Converges $\forall x$. (finite values of x).