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Section 5.3: Forced Motion.

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg - \beta\frac{dx}{dt} + f(t) \quad , \quad mg - ks = 0$$

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t)$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x + \frac{\beta}{m}\frac{dx}{dt} = \frac{f(t)}{m}$$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t) \quad , \quad F(t) = f(t)/m$$

This is a 2nd order linear non homogeneous ODE with constant coefficients.

Use either a) Method of undetermined coefficients

or b) Method of variation of parameters.

Ex:

Interpret + solve the initial value problem.

$$\frac{1}{5}\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t \qquad x(0) = 1/2 \ , \ x'(0) = 0$$

(1) $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 25\cos 4t \quad , \quad x(t) = x_H + x_p$

$= x_T + x_S$

The associated homogeneous equation

$$r^2 + 6r + 10 = 0 \qquad r = -3 \pm \sqrt{9 - 10}$$

$$= -3 \pm i$$

$$x_H(t) = e^{-3t}\left[C_1\cos t + C_2\sin t\right]$$

the method of undetermined coefficients: 3/16/05

$$x_p = A\cos 4t + B\sin 4t$$
$$x_p' = -4A\sin 4t + 4B\cos 4t$$
$$x_p'' = -16A\cos 4t - 16B\sin 4t$$

substitute in (1)

$$-16B\sin 4t + 6[-4A\sin 4t + 4B\cos 4t] + 10[A\cos 4t + B\sin 4t] = 25\cos 4t$$

$$(-16A + 24B + 10A)\cos 4t + (-16B - 24A + 10B)\sin 4t = 25\cos 4t$$

$$\rightarrow \quad -6A + 24B = 25 \qquad (2)$$
$$-24A - 6B = 0 \qquad (3)$$

$$\Rightarrow \quad -105B = -100 \quad \rightarrow B = \frac{50}{51}$$

$$(3) \rightarrow \quad A = \frac{6B}{-24} = 6 \times \frac{50}{51} \times \frac{1}{-24}$$

$$= \frac{-50}{204} = \frac{-25}{102}$$

$$\therefore x_p(t) = \frac{-25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

$$x_{total} = e^{-3t}[c_1\cos t + c_2\sin t] - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

$$x(0) = 1/2 \qquad c_1 - \frac{25}{102} = \frac{1}{2} \rightarrow c_1 = \frac{1}{2} + \frac{25}{102}$$
$$= \frac{51+25}{102} = \frac{76}{102}$$

$$c_1 = \frac{38}{51}$$

$$x'(t) = \quad \ldots t + c_2\cos t] + \frac{100}{102}\sin 4t + \frac{200}{51}\cos 4t$$

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$x'(0) = 0$

$$\Longrightarrow \quad -3\left[C_1\right] + 1.\left[C_2\right] + \frac{200}{51} = 0$$

$$C_2 = -\frac{200}{51} + 3C_1$$

$$C_2 = -\frac{200}{51} + \frac{3 \times 76}{102} = -\frac{200}{51} + \frac{3 \times 38}{51}$$

$$C_2 = \frac{-200 + 114}{51} = -\frac{86}{51}$$

$$\therefore \; x(t) = \underbrace{e^{-3t}\left[\frac{38}{51}\cos t - \frac{86}{51}\sin t\right]}_{\text{Transient Solution}} \underbrace{- \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t}_{\text{Steady State Solution.}}$$

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Ex2: Transient and Steady State Solutions

Solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 4\cos t + 2\sin t$, $x(0)=0$, $x'(0)=3$

The associated homogeneous equation is $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0$

$$\Rightarrow r^2+2r+2=0$$

$$r = -1\pm\sqrt{1-2} = -1\pm i$$

$\therefore x_H(t) = e^{-t}[C_1\cos t + C_2\sin t]$.

Let's use the method of undetermined coefficients:

$$x_p = A\cos t + B\sin t$$
$$x_p' = -A\sin t + B\cos t$$
$$x_p'' = -A\cos t - B\sin t$$

$$-A\cos t - B\sin t + 2(-A\sin t + B\cos t) + 2(A\cos t + B\sin t) = 4\cos t + 2\sin t$$

$$(-A+2B+2A)\cos t + (-B-2A+2B) = 4\cos t + 2\sin t$$

$$A+2B=4 \quad \text{— (1)}$$
$$-2A+B=2 \quad \text{— (2)}$$

$2\cdot(1)+(2)$ $\quad 5B=10 \Rightarrow \boxed{B=2}$, $\boxed{A=0}$

$$\therefore x_{total} = e^{-t}[C_1\cos t + C_2\sin t] + 2\sin t.$$

I.C.

$x(0)=0 \Rightarrow \boxed{C_1=0}$

$x'(t) = -e^{-t}[C_2\sin t] + C_2\cos t\, e^{-t} + 2\cos t$

$x'(0)=3 \Rightarrow + C_2 + 2 = 3 \Rightarrow \boxed{C_2=1}$

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$\Longrightarrow$ $x(t) = e^{-t} \sin t + 2 \sin t$

$x(t) = \sin t \left[2 + e^{-t}\right]$

or $x(t) = \underbrace{e^{-t} \sin t}_{\text{Transient Solution}} + \underbrace{2 \sin t}_{\text{Steady State Solution}}$

Ex 3 Forced Undamped Motion

Solve the IVP

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \gamma t \qquad x(0) = 0 \qquad x'(0) = 0$$

$$r^2 + \omega^2 = 0 \quad \Longrightarrow \quad r = \pm \omega i$$

$$x_H(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

$$x_p(t) = A \cos \gamma t + B \sin \gamma t$$

$$x_p'(t) = -\gamma A \sin \gamma t + \gamma B \cos \gamma t$$

$$x_p''(t) = -\gamma^2 A \cos \gamma t - \gamma^2 B \sin \gamma t$$

Substitute back in the D.E.

$$-\gamma^2 A \cos \gamma t - \gamma^2 B \sin \gamma t + \omega^2 (A \cos \gamma t + B \sin \gamma t) = F_0 \sin \gamma t$$

$$\Longrightarrow \left(-\gamma^2 A + \omega^2 A\right) \cos \gamma t + \left(-\gamma^2 B + \omega^2 B\right) = F_0 \sin \gamma t$$

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$$\Rightarrow \quad (-\gamma^2 A + \omega^2 A) = 0$$

$$A(\omega^2 - \gamma^2) = 0 \quad , \omega^2 \neq \gamma^2 \Longrightarrow A = 0$$

$$B(\omega^2 - \gamma^2) = F_0 \quad \Longrightarrow B = \frac{F_0}{(\omega^2 - \gamma^2)}$$

$$\Longrightarrow x_p(t) = \frac{F_0}{\omega^2 - \gamma^2} \sin \gamma t$$

$$\rightarrow x_{total}(t) = x_H + x_p$$

$$= C_1 \cos \omega t + C_2 \sin \omega t + \frac{F_0}{\omega^2 - \gamma^2} \sin \gamma t$$

$$x(0) = 0 \Longrightarrow C_1 = 0$$

$$x(t) = C_2 \sin \omega t + \frac{F_0}{(\omega^2 - \gamma^2)} \sin \gamma t$$

$$x'(t) = \omega C_2 \cos \omega t + \frac{F_0}{(\omega^2 - \gamma^2)} \gamma \cos \gamma t$$

$$x'(0) = 0 \Longrightarrow \omega C_2 + \frac{F_0}{(\omega^2 - \gamma^2)} \gamma = 0$$

$$C_2 = -\frac{\gamma F_0}{\omega(\omega^2 - \gamma^2)}$$

$$x(t) = \frac{-\gamma F_0}{\omega(\omega^2 - \gamma^2)} \sin \omega t + \frac{F_0}{(\omega^2 - \gamma^2)} \sin \gamma t \quad , \gamma \neq \omega.$$

$$x(t) = \frac{F_0}{\omega(\omega^2 - \gamma^2)} \left[-\gamma \sin \omega t + \omega \sin \gamma t \right] \quad ; \gamma \neq \omega$$

What Happens When $\gamma = \omega$.

ie Forcing Frequency = Natural Frequency

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ANSWER: Resonance Occurs 'Cause $\frac{0}{0}$

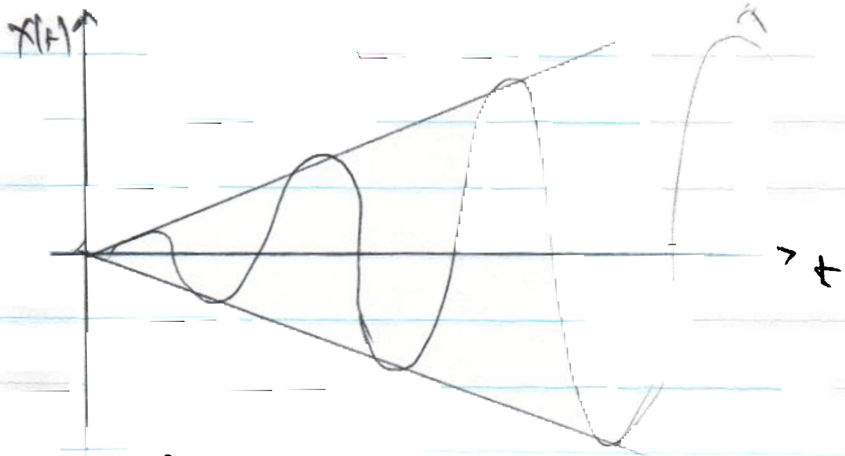
$$x(t) = \frac{F_0}{\omega(\omega^2-\gamma^2)}\left[-\gamma \sin\omega t + \omega \sin\gamma t\right]$$

$$\lim_{\gamma \to \omega} x(t) = F_0 \frac{\frac{d}{d\gamma}\left(-\gamma \sin\omega t + \omega \sin\gamma t\right)}{\frac{d}{d\gamma}\left[\omega(\omega^2-\gamma^2)\right]}$$

$$= \lim_{\gamma \to \omega} \frac{F_0\left(-\sin\omega t + t\omega \cos\gamma t\right)}{-2\gamma\omega}$$

$$= \frac{F_0\left(-\sin\omega t + t\omega\cos\omega t\right)}{-2\omega^2}$$

$$x(t) = \frac{F_0}{2\omega^2}\sin\omega t - \frac{F_0}{2\omega} t\cos\omega t$$



PURE RESONANCE

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Alternatively: I would like to solve the problem again when $\alpha = \omega$

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t \quad , \; x(0) = 0, \quad x'(0) = 0$$

$$s^2 + \omega^2 = 0 \longrightarrow s = \pm \omega i$$

$$x_H(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

$$x_p(t) = t\left[A \cos \omega t + B \sin \omega t\right]$$

$$x_p' = 1.\left[A \cos \omega t + B \sin \omega t\right] + t\left[-\omega A \sin \omega t + \omega B \cos \omega t\right]$$

$$x_p'' = -\omega A \sin \omega t + \omega B \cos \omega t + 1.\left[-\omega A \sin \omega t + \omega B \cos \omega t\right]$$
$$+ t\left[-\omega^2 A \cos \omega t - \omega^2 B \sin \omega t\right]$$

$$x_p'' = -2\omega A \sin \omega t + 2\omega B \cos \omega t - \omega^2 t\left[A \cos \omega t + B \sin \omega t\right]$$

substitute back in the ODE

$$\left\{\begin{array}{l} -2\omega A \sin \omega t + 2\omega B \cos \omega t - \omega^2 t\left[A \cos \omega t + B \sin \omega t\right] \\ + \omega^2 t\left[A \cos \omega t + B \sin \omega t\right] \end{array}\right\} = F_0 \sin \omega t$$

$$\rightarrow \quad -2\omega A \sin \omega t + 2\omega B \cos \omega t = F_0 \sin \omega t$$

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$$-2\omega A = F_0 \implies A = \frac{F_0}{-2\omega}$$

$$2\omega B = 0 \implies \boxed{B = 0}$$

$$\therefore X_p(t) = -\frac{F_0}{2\omega} t \cos\omega t$$

$$\therefore X_{total}(t) = C_1 \cos\omega t + C_2 \sin\omega t - \frac{F_0}{2\omega} t \cos\omega t$$

$$X(0) = 0 \implies \boxed{C_1 = 0}$$

$$X_{total}(t) = C_2 \sin\omega t - \frac{F_0}{2\omega} t \cos\omega t$$

$$X'(t) = \omega C_2 \cos\omega t - \frac{F_0}{2\omega}\left[1.\cos\omega t - \omega t \sin\omega t\right]$$

$$x'(0) = 0 \implies \omega C_2 = \frac{F_0}{2\omega} \implies \boxed{C_2 = \frac{F_0}{2\omega^2}}$$

$$\therefore X(t) = \frac{F_0}{2\omega^2} \sin\omega t - \frac{F_0}{2\omega} t \cos\omega t$$

which the same exact answer as $\lim_{\gamma \to \omega} X(t)$

YES.

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Resonance Curve: In the case of underdamped vibrations, the ODE is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F_0 \sin\gamma t \qquad \text{—(1)}$$

The associated homogeneous equation is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

$$r^2 + 2\lambda r + \omega^2 = 0$$

$$r = -\lambda \pm \sqrt{\lambda^2 - \omega^2} \quad ; \text{ but } \lambda < \omega$$

$$\Rightarrow \quad r = -\lambda \pm \sqrt{\omega^2 - \lambda^2}\, i$$

$$x_H(t) = e^{-\lambda t}\left[c_1 \cos\sqrt{\omega^2-\lambda^2}\, t + c_2 \sin\sqrt{\omega^2-\lambda^2}\, t\right]$$

$$x_p = A\cos\gamma t + B\sin\gamma t$$

$$x_p' = -\gamma A \sin\gamma t + \gamma B\cos\gamma t$$

$$x_p'' = -\gamma^2 A \cos\gamma t - \gamma^2 B \sin\gamma t$$

Substitute in (1)

$$\left[-\gamma^2 A\cos\gamma t - \gamma^2 B\sin\gamma t + 2\lambda\left(-\gamma A\sin\gamma t + \gamma B\cos\gamma t\right) + \omega^2\left(A\cos\gamma t + B\sin\gamma t\right)\right]$$

$$= F_0 \sin\gamma t$$

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$$\Rightarrow \left(-\gamma^2 A + 2\lambda\gamma B + \omega^2 A\right)\cos\gamma t + \left(-\gamma^2 B - 2\lambda\gamma A + \omega^2 B\right) = F_0 \sin\gamma t$$

$$\Rightarrow \begin{aligned} (\omega^2-\gamma^2)A + 2\lambda\gamma B &= 0 \\ -2\lambda\gamma A + (\omega^2-\gamma^2)B &= F_0 \end{aligned}$$

$$A = \frac{\begin{vmatrix} 0 & 2\lambda\gamma \\ F_0 & \omega^2-\gamma^2 \end{vmatrix}}{\begin{vmatrix} \omega^2-\gamma^2 & 2\lambda\gamma \\ -2\lambda\gamma & \omega^2-\gamma^2 \end{vmatrix}} = \frac{-2\lambda\gamma F_0}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}$$

$$B = \frac{\begin{vmatrix} \omega^2-\gamma^2 & 0 \\ -2\lambda\gamma & F_0 \end{vmatrix}}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2} = \frac{F_0(\omega^2-\gamma^2)}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}$$

$$x_{Total}(t) = e^{-\lambda t}\left[c_1\cos\sqrt{\omega^2-\lambda^2}\,t + c_2\sin\sqrt{\omega^2-\lambda^2}\,t\right] + A\cos\gamma t + B\sin\gamma t$$

Use the reduction formula.

$$A^2+B^2 = \frac{4\lambda^2\gamma^2F_0^2 + F_0^2(\omega^2-\gamma^2)^2}{\left[(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2\right]^2} = \frac{F_0^2\left[(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2\right]}{\left[(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2\right]^2}$$

$$A^2+B^2 = \frac{F_0^2}{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}$$

$$\sqrt{A^2+B^2} = \frac{F_0}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$$

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$$x(t) = Ce^{-\lambda t}\sin\left(\sqrt{\omega^2-\lambda^2}\ t+\phi\right) + \frac{F_0}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}\sin(\gamma t+\theta)$$

where $C = \sqrt{c_1^2 + c_2^2}$

$$\sin\phi = \frac{c_1}{c}, \quad \cos\phi = \frac{c_2}{c}$$

$$\sin\theta = \frac{A}{\sqrt{A^2+B^2}} = \frac{-2\lambda\gamma F_0}{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2} \cdot \frac{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}{F_0}$$

$$\sin\theta = \frac{-2\lambda\gamma}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$$

$$\cos\theta = \frac{B}{\sqrt{A^2+B^2}} = \frac{F_0(\omega^2-\gamma^2)}{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2} \cdot \frac{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}{F_0}$$

$$\cos\theta = \frac{(\omega^2-\gamma^2)}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$$

and I hereby verified the equations in the book on page 204.

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@ Steady Stat. $\lim_{t\to\infty} e^{-t} = 0$

$$x_{ss}(t) = \frac{F_0}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}} \sin(\gamma t + \theta)$$

where again $\sin\theta = \frac{-2\lambda\gamma}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$

$$\cos\theta = \frac{(\omega^2-\gamma^2)}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$$

$$|x(t)| \leq \frac{F_0}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$$

The maximum value of $\frac{F_0}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$

occurs when $\frac{d}{d\gamma}\left(\frac{F_0}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}\right) = 0$

$$\Rightarrow -\frac{F_0}{2}\left[(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2\right]^{-3/2}\left[2(\omega^2-\gamma^2)(-2\gamma)+8\lambda^2\gamma\right] = 0$$

$$\Rightarrow -4\gamma(\omega^2-\gamma^2)+8\lambda^2\gamma = 0$$

$$4\gamma\left[2\lambda^2-(\omega^2-\gamma^2)\right] = 0 \quad ; \gamma \neq 0$$

$$\Rightarrow 2\lambda^2-(\omega^2-\gamma^2) = 0$$

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$$2\lambda^2 - \omega^2 + \gamma^2 = 0$$

$$\gamma = \sqrt{\omega^2 - 2\lambda^2}$$

& maximum oscillations occur when the external force has a period $= \dfrac{2\pi}{\sqrt{\omega^2 - 2\lambda^2}}$

i.e a frequency $= \dfrac{\sqrt{\omega^2 - 2\lambda^2}}{2\pi}$

When this happens, the system is said to be in Resonance.