Section 6.3: 4/1/05

#10) $(x^2+1)y'' - 6y = 0$ —— (1)

Singular point are when $x^2+1=0$

$x = \pm i$

$\longrightarrow$ Radius of convergence $= |i| = 1$

R is at least $= 1$

$\longrightarrow$ let $Y = \sum_{n=0}^{\infty} c_n x^n \quad , |x| < 1$

$Y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad , Y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Substitute in (1)

$(x^2+1) \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 6 \sum_{n=0}^{\infty} c_n x^n = 0 \quad \forall x$

$\sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 6 \sum_{n=0}^{\infty} c_n x^n = 0 \quad \forall x$

$\sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - 6 \sum_{n=0}^{\infty} c_n x^n = 0$

$$\left\{ \begin{array}{l} 2c_2 + 6c_3 x - 6c_0 - 6c_1 x \\ + \sum_{n=2}^{\infty} \left[n(n-1)c_n + (n+2)(n+1)c_{n+2} - 6c_n \right] x^n \end{array} \right\} = 0$$

$\longrightarrow 2c_2 - 6c_0 = 0 \longrightarrow c_2 = \frac{6c_0}{2} = \boxed{3c_0}$

$6c_3 - 6c_1 = 0 \longrightarrow \boxed{c_3 = c_1}$

where c_0 and c_1 are arbitrary constants to be evaluated according to IVP.

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$$n(n-1)C_n + (n+2)(n+1)C_{n+2} - 6C_n = 0$$

$$C_{n+2} = \frac{6C_n - n(n-1)C_n}{(n+2)(n+1)} \quad ; n = 2, 3, \ldots$$

$$C_{n+2} \quad \frac{[6 - n(n-1)]C_n}{(n+2)(n+1)} \quad ; n = 2, 3, \ldots$$

$n=2$, $C_4 = \left(\frac{6 - 2 \cdot 1}{4 \cdot 3}\right) C_2 = \frac{C_2}{3} = \frac{3C_0}{3} = C_0$

$n=3$ $C_5 = \frac{(6 - 6)C_3}{5 \times 4} = 0$, $C_5 = 0$

$n=4$, $C_6 = \frac{[6 - 12]C_4}{6 \times 5} = -\frac{1}{5}C_4 = -\frac{1}{5}C_0$

$n=5$, $C_7 = (\quad)C_5 = 0$

$n=6$, $C_8 = \frac{[6 - 30]C_6}{8 \times 7} = \frac{-24}{8 \times 7}C_6 = -\frac{3}{7} \cdot -\frac{1}{5}C_0$

$= \frac{3}{35}C_0$

Notice that $C_3 = C_5 = C_7 = C_{2n+1} = C_{odd} = 0$

$\therefore y_{total} = C_0\left[1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \frac{3}{35}x^8 + \ldots\right]$
$+ C_1\left[x + x^3\right]$

$\therefore y_1 = 1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \frac{3}{35}x^8 + \ldots$
$y_2 = x + x^3$

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#10) $(x^2+1)y'' + 2xy' = 0 \quad , y(0)=0, \; y'(0)=1$

Singular Point are $x^2+1=0 \implies x = \pm i$

$R = |i| = 1 =$ Radius of Convergence

So $x=0$ is an ordinary point.

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1} \quad , \quad y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

Substitute

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + 2x \sum_{n=1}^{\infty} n C_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^n + \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + 2\sum_{n=1}^{\infty} n C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n + 2\sum_{n=1}^{\infty} n C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^n + 2C_2 + 6C_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) C_{n+2} x^n + 2C_1 x + 2\sum_{n=2}^{\infty} n C_n x^n = 0$$

$$2C_2 + (6C_3 + 2C_1)x + \sum_{n=2}^{\infty}\left[n(n-1)C_n + (n+2)(n+1)C_{n+2} + 2nC_n\right]x^n = 0 \quad , \forall x$$

$$\implies 2C_2 = 0 \implies \boxed{C_2 = 0}$$

$$6C_3 + 2C_1 = 0 \implies \boxed{C_3 = -\tfrac{1}{3}C_1}$$

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Recurrence Relation:

$$n(n-1)C_n + (n+2)(n+1)C_{n+2} + 2nC_n = 0 \qquad , n = 2, 3, \ldots$$

$$n^2 C_n - nC_n + (n+2)(n+1)C_{n+2} + 2nC_n = 0$$

$$\Rightarrow C_{n+2} = \frac{-(n^2+n)C_n}{(n+2)(n+1)}$$

$$C_{n+2} = -\frac{n}{(n+2)}C_n \quad , n = 2, 3, 4 \ldots$$

So far, we know that $C_2 = 0$, $C_3 = -\frac{1}{3}C_1$

$n=2$, $\quad C_4 = -\frac{2}{4}C_2 = 0.$

$n=3$, $\quad C_5 = -\frac{3}{5}C_3 = -\frac{3}{5}\cdot -\frac{1}{3}C_1 = \frac{1}{5}C_1$

$n=4$, $\quad C_6 = 0$, $\quad C_{EVEN} = 0$.

$n=5$, $\quad C_7 = -\frac{5}{7}C_5 = -\frac{5}{7}\cdot\frac{1}{5}C_1 = -\frac{1}{7}C_1$

$\therefore y_{total} = C_0 + C_1\left[x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right]$

I.C. $y(0) = 0 \Rightarrow \boxed{C_0 = 0}$

$$y'(x) = C_1\left[1 - x^2 + x^4 - x^6 + \ldots\right]$$

$y'(0) = 1 \Rightarrow \boxed{C_1 = 1}$, $\therefore y_{total} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$

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$$y_{total} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{(2n-1)}}{(2n-1)}$$

$$y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{(2n-1)}}{(2n-1)}$$