Section 6.1 3/28/05

#12) $x^2y'' + 8xy' + 6y = 0$

$r^2 + (8-1)r + 6 = 0$

$r^2 + 7r + 6 = 0$

$(r+1)(r+6) = 0 \implies r = -1, -6$

$\therefore y_H = c_1x^{-1} + c_2x^{-6}$

#26) $x^2y'' - 3xy' + 4y = 0$, $y(1) = 5$, $y'(1) = 3$

$r^2 + (-3-1)r + 4 = 0$

$r^2 - 4r + 4 = 0$

$r = 2 \pm \sqrt{4-4} = 2$

$\therefore y_H(x) = c_1x^2 + c_2x^2 \ln x$

$y' = 2c_1x + 2c_2x\ln x + c_2x^2 \cdot \frac{1}{x}$
$= 2c_1x + 2c_2x\ln x + c_2x$

$y'' = 2c_1 + c_2 + 2c_2[\ln x + 1]$ (We don't need y''!)

$y(1) = 5 \implies \boxed{c_1 = 5}$
$y'(1) = 3 \implies 2c_1 + c_2 = 3$
$\implies c_2 = 3 - 2c_1 = -7$, $\boxed{c_2 = -7}$

$\therefore y_H(x) = 5x^2 - 7x^2\ln|x|$

Section 6.2 3/31/05

#28)
$$y'' - y = 0 \qquad \text{— (1)}$$
$$r^2 - 1 = 0 \qquad \Longrightarrow r = \pm 1$$

$$\therefore y(x) = c_1 e^{-x} + c_2 e^{x}$$

Now let us solve by the series solution method.

Assume that $y = \sum_{n=0}^{\infty} c_n x^n$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad , \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substitute in (1)

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} - c_n \right] x^n = 0 \qquad \forall x$$

$$\longrightarrow (n+2)(n+1) c_{n+2} - c_n = 0$$

$$\therefore c_{n+2} = \frac{c_n}{(n+2)(n+1)} \quad , \quad c_0, c_1$$

$n=0$, $\quad c_2 = \frac{c_0}{2} = \frac{c_0}{2!}$

$n=1$, $\quad c_3 = \frac{c_1}{6} = \frac{c_1}{3!}$

$n=2$ $\quad c_4 = \frac{c_2}{12} = \frac{c_0}{2 \times 12} = \frac{c_0}{4!}$

3/31/05

$n=3, \quad C_5 = \frac{C_3}{15} = \frac{C_1}{6 \times 15} = \frac{C_1}{5!}$

$\therefore \quad C_{2n} = \frac{C_0}{(2n)!}$

$C_{2n+1} = \frac{C_1}{(2n+1)!}$

$\therefore \quad y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 - - -$

$y = C_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + C_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

Recall that $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$

$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad$ ~~at~~ $N = 2n \quad \rightarrow$

$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \sum_{N=0}^{\infty} \frac{x^N}{N!} = e^x$

$y = C_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + C_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$= C_0 \cosh x + C_1 \sinh x$

$= C_0 \left(\frac{e^x + e^{-x}}{2}\right) + C_1 \left(\frac{e^x - e^{-x}}{2}\right)$

$y = \left(\frac{C_0 + C_1}{2}\right) e^x + \left(\frac{C_0 - C_1}{2}\right) e^{-x} \quad$ Verified

$= C_1 e^x + C_2 e^{-x}$