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Section 3.1

#16) $y^2 - x^2 = c_1 x^3 \implies c_1 = \frac{y^2 - x^2}{x^3}$

$2y\,dy - 2x\,dx = 3c_1 x^2 dx$

$2y\,dy = (3c_1 x^2 + 2x)dx$

$\frac{dy}{dx} = \frac{(3c_1 x^2 + 2x)}{2y} = \frac{\left(3(y^2 - x^2)/x + 2x\right)}{2y}$

$\frac{dy}{dx} = \frac{3(y^2 - x^2) + 2x^2}{2xy}$

$= \frac{3y^2 - x^2}{2xy}$

So for the orthogonal trajectory

$\frac{dy}{dx} = \frac{-2xy}{3y^2 - x^2}$ Homogeneous Equation —(1)

$\implies +2xy\,dx + (3y^2 - x^2)dy = 0$

$\frac{\partial(2xy)}{\partial y} = 2x \neq \frac{\partial(3y^2 - x^2)}{\partial x} = -2x$

$\implies$ Eq. Is not exact.

Since the eq. is Homogeneous.

let $y = u(x) \cdot x$

$y' = u'x + u$

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Substitute into eq. (1)

$$u'x+u = \frac{-2x\,ux}{3u^2x^2-x^2} = \frac{-2u}{3u^2-1} \quad ; \; x \neq 0$$

$$u'x = \frac{-2u}{3u^2-1} - u = \frac{-2u-u(3u^2-1)}{3u^2-1}$$

$$= \frac{-2u-3u^3+u}{(3u^2-1)}$$

$$u'x = \frac{-(3u^3+u)}{(3u^2-1)}$$

$$\int \frac{(1-3u^2)\,du}{(3u^3+u)} = \int \frac{dx}{x} \longleftrightarrow (2)$$

Now, integrate by partial fractions.

$$\frac{1-3u^2}{u(3u^2+1)} = \frac{A}{u} + \frac{Bu+C}{3u^2+1}$$

$$A = \left.\frac{1-3u^2}{3u^2+1}\right|_{u=0} = 1$$

$$\frac{1-3u^2}{u(3u^2+1)} = \frac{1}{u} + \frac{Bu+C}{3u^2+1} = \frac{3u^2+1+u(Bu+C)}{u(3u^2+1)}$$

$$= \frac{(3+B)u^2+Cu+1}{u(3u^2+1)}$$

Now match the coefficients

$$C=0 \quad , \quad 3+B=-3 \quad \Longrightarrow \boxed{B=-6}$$

$$(2) \Rightarrow \int\left(\frac{1}{u} \; \frac{-6u}{3u^2+1}\right)du = \int \frac{dx}{x}$$

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$$\ln|u| - \ln|3u^2+1| = \ln|x| + \ln c$$

$$\ln\left|\frac{u}{c(3u^2+1)}\right| = \ln|x|$$

$$\frac{u}{3u^2+1} = cx$$

$$\frac{y/x}{3(y/x)^2+1} = cx$$

$$\frac{yx}{3y^2+x^2} = cx \qquad \text{; } x \neq 0$$

$$\boxed{y = c(3y^2+x^2)}$$

#28) $3xy^2 = 2 + 3c_1 x \longrightarrow c_1 = (3xy^2-2)/3x$

$$3\left[1 \cdot y^2 + 2xy\frac{dy}{dx}\right] = 3c_1$$

$$2xy\frac{dy}{dx} = c_1 - y^2 = \frac{3xy^2 - 2 - 3xy^2}{3x}$$

$$\frac{dy}{dx} = \frac{-2}{3x} \cdot \frac{1}{2xy} = -\frac{1}{3x^2y}$$

Orthogonal Trajectory $\Longrightarrow \dfrac{dy}{dx} = 3x^2y$

$$\int \frac{dy}{y} = \int 3x^2\,dx$$

$$\ln|y| = x^3 + \ln c$$

$$y = ce^{x^3}$$

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Given that the orthogonal trajectory must pass thru $(0,10)$

$$y = c e^{x^3}$$

$$10 = c$$

$$\boxed{\therefore\ y = 10 e^{x^3}}$$

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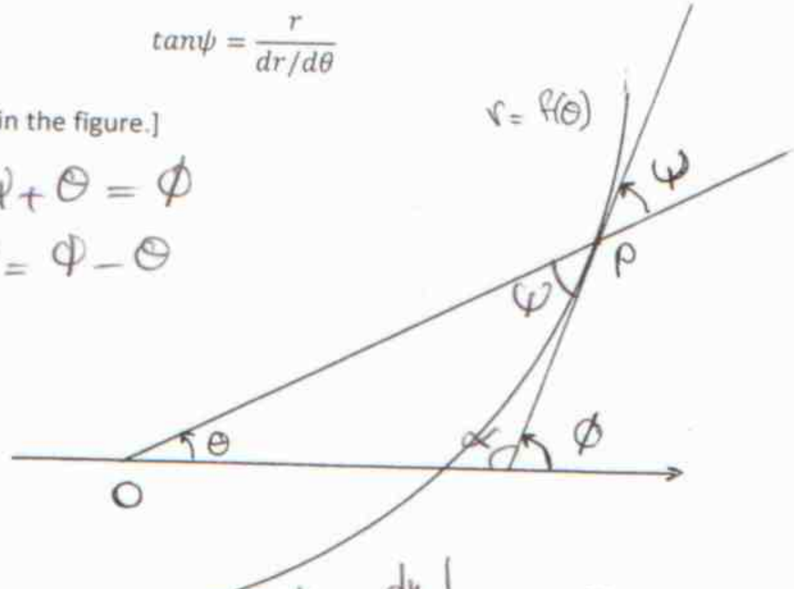
Special Assignment

Let P be any point (except the origin) on the curve $r = f(\theta)$. *If* ψ is the angle between the tangent line at P and the radial line OP, show that

$$tan\psi = \frac{r}{dr/d\theta}$$

{Hint: Observe that $\psi = \phi - \theta$ in the figure.]



$$\left.\begin{array}{l}\psi + \theta + \alpha = \pi \\ \phi + \alpha = \pi\end{array}\right\} \Rightarrow \begin{array}{l}\psi + \theta = \phi \\ \psi = \phi - \theta\end{array}$$

$$\tan\psi = \tan(\phi - \theta)$$

$$= \frac{\tan\phi - \tan\theta}{1 + \tan\phi . \tan\theta} \quad ; \text{ But } \tan\phi = \left.\frac{dy}{dx}\right|_{@\text{ Point } P}$$

$$= \frac{\frac{dy}{dx} - \tan\theta}{1 + dy/dx . \tan\theta} = \frac{\frac{dy/d\theta}{dx/d\theta} - \tan\theta}{1 + \frac{dy/d\theta}{dx/d\theta} . \tan\theta} \quad \text{—— (1)}$$

Now $x = r\cos\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta$

$y = r\sin\theta \Rightarrow \frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta$

$$\text{Eq.}(1) \Rightarrow \tan\psi = \frac{\frac{dy}{d\theta} - \frac{dx}{d\theta} . \tan\theta}{\frac{dx}{d\theta} + \frac{dy}{d\theta} . \tan\theta} \quad ; \frac{dx}{d\theta} \neq 0$$

$$\tan\psi = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta - \left(\cos\theta\frac{dr}{d\theta} - r\sin\theta\right) . \tan\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta + \left(\sin\theta\frac{dr}{d\theta} + r\cos\theta\right) . \tan\theta}$$

$$= \frac{r\cos\theta + r\sin^2\theta/\cos\theta}{\cos\theta\frac{dr}{d\theta} + \frac{\sin^2\theta}{\cos\theta} . \frac{dr}{d\theta}} = \frac{r(\cos^2\theta + \sin^2\theta)}{(\cos^2\theta + \sin^2\theta)\frac{dr}{d\theta}} \quad ; \cos\theta \neq 0$$

$$\therefore \tan\psi = \frac{r}{dr/d\theta} \quad \text{Q.E.D}$$