9.2 Area of a Surface of Revolution

1. $y = \ln x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (1/x)^2}\,dx \;\Rightarrow\; S = \int_1^3 2\pi(\ln x)\sqrt{1 + (1/x)^2}\,dx$ [by (7)]

2. $y = \sin^2 x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (2\sin x\cos x)^2}\,dx \;\Rightarrow$

$S = \int_0^{\pi/2} 2\pi\sin^2 x\sqrt{1 + (2\sin x\cos x)^2}\,dx$ [by (7)]

3. $y = \sec x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (\sec x\tan x)^2}\,dx \;\Rightarrow$

$S = \int_0^{\pi/4} 2\pi x\sqrt{1 + (\sec x\tan x)^2}\,dx$ [by (8)]

4. $y = e^x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + e^{2x}}\,dx \;\Rightarrow\; S = \int_0^{\ln 2} 2\pi x\sqrt{1 + e^{2x}}\,dx$ [by (8)] or

$\int_1^2 2\pi(\ln y)\sqrt{1 + (1/y)^2}\,dy$ [by (6)]

5. $y = x^3 \;\Rightarrow\; y' = 3x^2$. So

$$S = \int_0^2 2\pi y\sqrt{1 + (y')^2}\,dx = 2\pi\int_0^2 x^3\sqrt{1 + 9x^4}\,dx \qquad [u = 1 + 9x^4,\ du = 36x^3\,dx]$$

$$= \tfrac{2\pi}{36}\int_1^{145}\sqrt{u}\,du = \tfrac{\pi}{18}\left[\tfrac{2}{3}u^{3/2}\right]_1^{145} = \tfrac{\pi}{27}\left(145\sqrt{145} - 1\right)$$

6. The curve $9x = y^2 + 18$ is symmetric about the x-axis, so we only use its top half, given by

$y = 3\sqrt{x - 2}$. $dy/dx = \dfrac{3}{2\sqrt{x-2}}$, so $1 + (dy/dx)^2 = 1 + \dfrac{9}{4(x-2)}$. Thus,

$$= \int_2^6 2\pi\cdot 3\sqrt{x-2}\sqrt{1 + \frac{9}{4(x-2)}}\,dx = 6\pi\int_2^6\sqrt{x - 2 + \tfrac{9}{4}}\,dx = 6\pi\int_2^6\left(x + \tfrac{1}{4}\right)^{1/2}dx$$

$$= 6\pi\cdot\tfrac{2}{3}\left[\left(x + \tfrac{1}{4}\right)^{3/2}\right]_2^6 = 4\pi\left[\left(\tfrac{25}{4}\right)^{3/2} - \left(\tfrac{9}{4}\right)^{3/2}\right] = 4\pi\left(\tfrac{125}{8} - \tfrac{27}{8}\right) = 4\pi\cdot\tfrac{98}{8} = 49\pi$$

7. $y = \sqrt{x} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + [1/(2\sqrt{x})]^2 = 1 + 1/(4x)$. So

$$S = \int_4^9 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_4^9 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx = 2\pi\int_4^9\sqrt{x + \tfrac{1}{4}}\,dx$$

$$= 2\pi\left[\tfrac{2}{3}\left(x + \tfrac{1}{4}\right)^{3/2}\right]_4^9 = \tfrac{4\pi}{3}\left[\tfrac{1}{8}(4x + 1)^{3/2}\right]_4^9 = \tfrac{\pi}{6}\left(37\sqrt{37} - 17\sqrt{17}\right)$$

8. $y = \cos 2x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (-2\sin 2x)^2}\,dx \;\Rightarrow$

$$S = \int_0^{\pi/6} 2\pi\cos 2x\sqrt{1 + 4\sin^2 2x}\,dx = 2\pi\int_0^{\sqrt{3}}\sqrt{1 + u^2}\left(\tfrac{1}{4}\,du\right) \qquad [u = 2\sin 2x,\ du = 4\cos 2x\,dx]$$

$$\overset{21}{=} \tfrac{\pi}{2}\left[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^{\sqrt{3}} = \tfrac{\pi}{2}\left[\tfrac{\sqrt{3}}{2}\cdot 2 + \tfrac{1}{2}\ln\left(\sqrt{3} + 2\right)\right] = \tfrac{\pi\sqrt{3}}{2} + \tfrac{\pi}{4}\ln\left(2 + \sqrt{3}\right)$$

9. $y = \cosh x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \sinh^2 x = \cosh^2 x$. So

$$S = 2\pi \int_0^1 \cosh x \cosh x \, dx = 2\pi \int_0^1 \tfrac{1}{2}(1 + \cosh 2x)\, dx = \pi\left[x + \tfrac{1}{2}\sinh 2x\right]_0^1$$

$$= \pi\left(1 + \tfrac{1}{2}\sinh 2\right) \quad \text{or} \quad \pi\left[1 + \tfrac{1}{4}\left(e^2 - e^{-2}\right)\right]$$

10. $y = \dfrac{x^3}{6} + \dfrac{1}{2x} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{x^2}{2} - \dfrac{1}{2x^2} \;\Rightarrow$

$$\sqrt{1 + (dy/dx)^2} = \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} = \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} = \frac{x^2}{2} + \frac{1}{2x^2} \;\Rightarrow$$

$$S = \int_{1/2}^1 2\pi\left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = 2\pi\int_{1/2}^1 \left(\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3}\right) dx$$

$$= 2\pi\int_{1/2}^1 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{x^{-3}}{4}\right) dx = 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{x^{-2}}{8}\right]_{1/2}^1$$

$$= 2\pi\left[\left(\tfrac{1}{72} + \tfrac{1}{6} - \tfrac{1}{8}\right) - \left(\tfrac{1}{64 \cdot 72} + \tfrac{1}{24} - \tfrac{1}{2}\right)\right] = 2\pi\left(\tfrac{263}{512}\right) = \tfrac{263}{256}\pi$$

11. $x = \frac{1}{3}(y^2 + 2)^{3/2} \;\Rightarrow\; dx/dy = \frac{1}{2}(y^2 + 2)^{1/2}(2y) = y\sqrt{y^2 + 2} \;\Rightarrow$

$1 + (dx/dy)^2 = 1 + y^2(y^2 + 2) = (y^2 + 1)^2$. So

$$S = 2\pi\int_1^2 y(y^2 + 1)\, dy = 2\pi\left[\tfrac{1}{4}y^4 + \tfrac{1}{2}y^2\right]_1^2 = 2\pi\left(4 + 2 - \tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{21\pi}{2}$$

12. $x = 1 + 2y^2 \;\Rightarrow\; 1 + (dx/dy)^2 = 1 + (4y)^2 = 1 + 16y^2$. So

$$S = 2\pi\int_1^2 y\sqrt{1 + 16y^2}\, dy = \tfrac{\pi}{16}\int_1^2 \left(16y^2 + 1\right)^{1/2} 32y\, dy = \tfrac{\pi}{16}\left[\tfrac{2}{3}\left(16y^2 + 1\right)^{3/2}\right]_1^2$$

$$= \tfrac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)$$

13. $y = \sqrt[3]{x} \;\Rightarrow\; x = y^3 \;\Rightarrow\; 1 + (dx/dy)^2 = 1 + 9y^4$. So

$$S = 2\pi\int_1^2 x\sqrt{1 + (dx/dy)^2}\, dy = 2\pi\int_1^2 y^3\sqrt{1 + 9y^4}\, dy = \tfrac{2\pi}{36}\int_1^2 \sqrt{1 + 9y^4}\, 36y^3\, dy$$

$$= \tfrac{\pi}{18}\left[\tfrac{2}{3}\left(1 + 9y^4\right)^{3/2}\right]_1^2 = \tfrac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right)$$

14. $y = 1 - x^2 \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + 4x^2 \;\Rightarrow$

$$S = 2\pi\int_0^1 x\sqrt{1 + 4x^2}\, dx = \tfrac{\pi}{4}\int_0^1 8x\sqrt{4x^2 + 1}\, dx = \tfrac{\pi}{4}\left[\tfrac{2}{3}\left(4x^2 + 1\right)^{3/2}\right]_0^1 = \tfrac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

15. $x = \sqrt{a^2 - y^2} \;\Rightarrow\; dx/dy = \frac{1}{2}(a^2 - y^2)^{-1/2}(-2y) = -y/\sqrt{a^2 - y^2} \;\Rightarrow$

$$1 + (dx/dy)^2 = 1 + \frac{y^2}{a^2 - y^2} = \frac{a^2 - y^2}{a^2 - y^2} + \frac{y^2}{a^2 - y^2} = \frac{a^2}{a^2 - y^2} \;\Rightarrow$$

$S = \displaystyle\int_0^{a/2} 2\pi\sqrt{a^2 - y^2}\,\frac{a}{\sqrt{a^2 - y^2}}\, dy = 2\pi\int_0^{a/2} a\, dy = 2\pi a\left[y\right]_0^{a/2} = 2\pi a\left(\frac{a}{2} - 0\right) = \pi a^2$. Note that this

$\frac{1}{4}$ the surface area of a sphere of radius a, and the length of the interval $y = 0$ to $y = a/2$ is $\frac{1}{4}$ the length of the interval $y = -a$ to $y = a$.

$x = a\cosh(y/a) \quad\Rightarrow\quad 1 + (dx/dy)^2 = 1 + \sinh^2(y/a) = \cosh^2(y/a)$. So

$$S = 2\pi \int_{-a}^{a} a\cosh\left(\frac{y}{a}\right)\cosh\left(\frac{y}{a}\right) dy = 4\pi a \int_0^a \cosh^2\left(\frac{y}{a}\right) dy = 2\pi a \int_0^a \left[1 + \cosh\left(\frac{2y}{a}\right)\right] dy$$

$$= 2\pi a\left[y + \frac{a}{2}\sinh\left(\frac{2y}{a}\right)\right]_0^a = 2\pi a\left[a + \frac{a}{2}\sinh 2\right] = 2\pi a^2\left[1 + \tfrac{1}{2}\sinh 2\right] \text{ or } \frac{\pi a^2(e^2 + 4 - e^{-2})}{2}$$

. $y = \ln x \quad\Rightarrow\quad dy/dx = 1/x \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + 1/x^2 \quad\Rightarrow\quad S = \int_1^3 2\pi \ln x\sqrt{1 + 1/x^2}\,dx$.

Let $f(x) = \ln x\sqrt{1 + 1/x^2}$. Since $n = 10$, $\Delta x = \frac{3-1}{10} = \frac{1}{5}$. Then

$S \approx S_{10} = 2\pi \cdot \frac{1/5}{3}\,[f(1) + 4f(1.2) + 2f(1.4) + \cdots + 2f(2.6) + 4f(2.8) + f(3)] \approx 9.023754$.

The value of the integral produced by a calculator is 9.024262 (to six decimal places).

18. $y = x + \sqrt{x} \quad\Rightarrow\quad dy/dx = 1 + \frac{1}{2}x^{-1/2} \quad\Rightarrow\quad 1 + (dy/dx)^2 = 2 + x^{-1/2} + \frac{1}{4}x^{-1} \quad\Rightarrow$

$S = \int_1^2 2\pi(x + \sqrt{x})\sqrt{2 + \frac{1}{\sqrt{x}} + \frac{1}{4x}}\,dx$. Let $f(x) = (x + \sqrt{x})\sqrt{2 + \frac{1}{\sqrt{x}} + \frac{1}{4x}}$.

Since $n = 10$, $\Delta x = \frac{2-1}{10} = \frac{1}{10}$. Then

$S \approx S_{10} = 2\pi \cdot \frac{1/10}{3}\,[f(1) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] \approx 29.506566$.

The value of the integral produced by a calculator is 29.506568 (to six decimal places).

$y = \sec x \quad\Rightarrow\quad dy/dx = \sec x\tan x \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \sec^2 x\tan^2 x \quad\Rightarrow$

$S = \int_0^{\pi/3} 2\pi\sec x\sqrt{1 + \sec^2 x\tan^2 x}\,dx$. Let $f(x) = \sec x\sqrt{1 + \sec^2 x\tan^2 x}$.

Since $n = 10$, $\Delta x = \dfrac{\pi/3 - 0}{10} = \dfrac{\pi}{30}$. Then

$$S \approx S_{10} = 2\pi \cdot \frac{\pi/30}{3}\left[f(0) + 4f\left(\frac{\pi}{30}\right) + 2f\left(\frac{2\pi}{30}\right) + \cdots + 2f\left(\frac{8\pi}{30}\right) + 4f\left(\frac{9\pi}{30}\right) + f\left(\frac{\pi}{3}\right)\right] \approx 13.527296.$$

The value of the integral produced by a calculator is 13.516987 (to six decimal places).

$$y = (1 + e^x)^{1/2} \quad\Rightarrow\quad \frac{dy}{dx} = \frac{1}{2}(1 + e^x)^{-1/2} \cdot e^x = \frac{e^x}{2(1 + e^x)^{1/2}} \quad\Rightarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{e^{2x}}{4(1 + e^x)} = \frac{4 + 4e^x + e^{2x}}{4(1 + e^x)} = \frac{(e^x + 2)^2}{4(1 + e^x)} \quad\Rightarrow$$

$$S = \int_0^1 2\pi\sqrt{1 + e^x}\,\frac{e^x + 2}{2\sqrt{1 + e^x}}\,dx = \pi\int_0^1 (e^x + 2)\,dx = \pi\left[e^x + 2x\right]_0^1 = \pi[(e + 2) - (1 + 0)] = \pi(e + 1).$$

Let $f(x) = \frac{1}{2}(e^x + 2)$. Since $n = 10$, $\Delta x = \frac{1-0}{10} = \frac{1}{10}$. Then

$S \approx S_{10} = 2\pi \cdot \frac{1/10}{3}\,[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1)] \approx 11.681330$.

The value of the integral produced by a calculator is 11.681327 (to six decimal places).