9 □ FURTHER APPLICATIONS OF INTEGRATION

9.1 Arc Length

1. $y = 2 - 3x \Rightarrow L = \int_{-2}^{1} \sqrt{1 + (dy/dx)^2}\,dx = \int_{-2}^{1} \sqrt{1 + (-3)^2}\,dx = \sqrt{10}\,[1 - (-2)] = 3\sqrt{10}.$

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$$L = [\text{distance from } (-2, 8) \text{ to } (1, -1)] = \sqrt{[1 - (-2)]^2 + [(-1) - 8]^2} = \sqrt{90} = 3\sqrt{10}$$

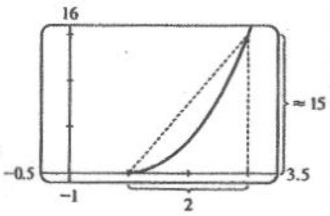
2. Using the arc length formula with $y = \sqrt{4 - x^2} \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{\sqrt{4 - x^2}}$, we get

$$L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{1 + \frac{x^2}{4 - x^2}}\,dx = \int_0^2 \frac{2\,dx}{\sqrt{4 - x^2}} = 2 \lim_{t \to 2^-} \int_0^t \frac{dx}{\sqrt{2^2 - x^2}}$$

$$= 2 \lim_{t \to 2^-} \left[\sin^{-1}(x/2)\right]_0^t = 2 \lim_{t \to 2^-} \left[\sin^{-1}(t/2) - \sin^{-1} 0\right] = 2\left(\tfrac{\pi}{2} - 0\right) = \pi$$

The curve is a quarter of a circle with radius 2, so the length of the arc is $\frac{1}{4}(2\pi \cdot 2) = \pi$, as above.

3.

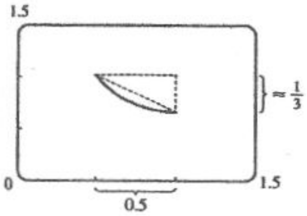


From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1, 0)$, $(3, 0)$, and $(3, f(3)) \approx (3, 15)$, where $y = f(x) = \frac{2}{3}(x^2 - 1)^{3/2}$. This length is about $\sqrt{15^2 + 2^2} \approx 15$, so we might estimate the length to be 15.5. $y = \frac{2}{3}(x^2 - 1)^{3/2} \Rightarrow y' = (x^2 - 1)^{1/2}(2x) \Rightarrow$ $1 + (y')^2 = 1 + 4x^2(x^2 - 1) = 4x^4 - 4x^2 + 1 = (2x^2 - 1)^2$, so, using the fact that $2x^2 - 1 > 0$ for $1 \le x \le 3$,

$$L = \int_1^3 \sqrt{(2x^2 - 1)^2}\,dx = \int_1^3 \left|2x^2 - 1\right| dx = \int_1^3 (2x^2 - 1)\,dx = \left[\tfrac{2}{3}x^3 - x\right]_1^3$$

$$= (18 - 3) - \left(\tfrac{2}{3} - 1\right) = \tfrac{46}{3} = 15.\overline{3}$$

4.



From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(0.5, f(0.5) \approx 1)$, $(1, f(0.5) \approx 1)$ and $\left(1, \frac{2}{3}\right)$, where $y = f(x) = x^3/6 + 1/(2x)$. This length is about $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2} \approx 0.6$, so we might estimate the length to be 0.65.

$$y = \frac{x^3}{6} + \frac{1}{2x} \Rightarrow y' = \frac{x^2}{2} - \frac{x^{-2}}{2} \Rightarrow$$

$$1 + (y')^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{x^{-4}}{4} = \frac{x^4}{4} + \frac{1}{2} + \frac{x^{-4}}{4} = \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2$$

so, using the fact that the parenthetical expression is positive,

$$L = \int_{1/2}^{1} \sqrt{\left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2}\,dx = \int_{1/2}^{1} \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_{1/2}^{1}$$

$$= \left(\tfrac{1}{6} - \tfrac{1}{2}\right) - \left(\tfrac{1}{48} - 1\right) = \tfrac{31}{48} = 0.6458\overline{3}$$

5. $y = 1 + 6x^{3/2} \Rightarrow dy/dx = 9x^{1/2} \Rightarrow 1 + (dy/dx)^2 = 1 + 81x$. So

$L = \int_0^1 \sqrt{1+81x}\,dx = \int_1^{82} u^{1/2}\left(\frac{1}{81}\,du\right)$ [where $u = 1 + 81x$ and $du = 81\,dx$]

$= \frac{1}{81}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^{82} = \frac{2}{243}\left(82\sqrt{82} - 1\right)$

6. $y^2 = 4(x+4)^3, y > 0 \Rightarrow y = 2(x+4)^{3/2} \Rightarrow dy/dx = 3(x+4)^{1/2} \Rightarrow$

$1 + (dy/dx)^2 = 1 + 9(x+4) = 9x + 37$. So

$$L = \int_0^2 \sqrt{9x+37}\,dx \quad \begin{bmatrix} u = 9x+37, \\ du = 9\,dx \end{bmatrix} \quad = \int_{37}^{55} u^{1/2}\left(\frac{1}{9}\,du\right)$$

$$= \frac{1}{9}\cdot\frac{2}{3}\left[u^{3/2}\right]_{37}^{55} = \frac{2}{27}\left(55\sqrt{55} - 37\sqrt{37}\right)$$

7. $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3} \Rightarrow \dfrac{dy}{dx} = \dfrac{5}{6}x^4 - \dfrac{3}{10}x^{-4} \Rightarrow$

$1 + (dy/dx)^2 = 1 + \frac{25}{36}x^8 - \frac{1}{2} + \frac{9}{100}x^{-8} = \frac{25}{36}x^8 + \frac{1}{2} + \frac{9}{100}x^{-8} = \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2$. So

$$L = \int_1^2 \sqrt{\left(\tfrac{5}{6}x^4 + \tfrac{3}{10}x^{-4}\right)^2}\,dx = \int_1^2 \left(\tfrac{5}{6}x^4 + \tfrac{3}{10}x^{-4}\right)dx = \left[\tfrac{1}{6}x^5 - \tfrac{1}{10}x^{-3}\right]_1^2$$

$$= \left(\tfrac{32}{6} - \tfrac{1}{80}\right) - \left(\tfrac{1}{6} - \tfrac{1}{10}\right) = \tfrac{31}{6} + \tfrac{7}{80} = \tfrac{1261}{240}$$

8. $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4} \Rightarrow \dfrac{dy}{dx} = x - \dfrac{1}{4x} \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = x^2 + \dfrac{1}{2} + \dfrac{1}{16x^2}$. So

$$L = \int_2^4 \left(x + \frac{1}{4x}\right)dx = \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_2^4 = \left(8 + \frac{2\ln 2}{4}\right) - \left(2 + \frac{\ln 2}{4}\right) = 6 + \frac{\ln 2}{4}.$$

9. $x = \frac{1}{3}\sqrt{y}\,(y-3) = \frac{1}{3}y^{3/2} - y^{1/2} \Rightarrow dx/dy = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2$. So

$$L = \int_1^9 \left(\tfrac{1}{2}y^{1/2} + \tfrac{1}{2}y^{-1/2}\right)dy = \tfrac{1}{2}\left[\tfrac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \tfrac{1}{2}\left[\left(\tfrac{2}{3}\cdot 27 + 2\cdot 3\right) - \left(\tfrac{2}{3}\cdot 1 + 2\cdot 1\right)\right]$$

$$= \tfrac{1}{2}\left(24 - \tfrac{8}{3}\right) = \tfrac{1}{2}\left(\tfrac{64}{3}\right) = \tfrac{32}{3}$$

10. $y = \ln(\cos x) \Rightarrow dy/dx = -\tan x \Rightarrow 1 + (dy/dx)^2 = 1 + \tan^2 x = \sec^2 x$. So

$$L = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/3} \sec x\,dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/3} = \ln\left(2 + \sqrt{3}\right) - \ln(1+0) = \ln\left(2 + \sqrt{3}\right)$$

11. $y = \ln(\sec x) \Rightarrow \dfrac{dy}{dx} = \dfrac{\sec x \tan x}{\sec x} = \tan x \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$, so

$$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} |\sec x|\,dx = \int_0^{\pi/4} \sec x\,dx = \left[\ln(\sec x + \tan x)\right]_0^{\pi/4}$$

$$= \ln\left(\sqrt{2} + 1\right) - \ln(1+0) = \ln\left(\sqrt{2} + 1\right)$$

12. $y = \ln x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} \Rightarrow \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + \left(\dfrac{1}{x}\right)^2} = \dfrac{\sqrt{1+x^2}}{x}$. So $L = \displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x}\,dx$. Now

let $v = \sqrt{1+x^2}$, so $v^2 = 1 + x^2$ and $v\,dv = x\,dx$. Thus

$$L = \int_{\sqrt{2}}^2 \frac{v}{v^2 - 1}v\,dv = \int_{\sqrt{2}}^2 \left(1 + \frac{1/2}{v-1} - \frac{1/2}{v+1}\right)dv = \left[v + \tfrac{1}{2}\ln|v-1| - \tfrac{1}{2}\ln|v+1|\right]_{\sqrt{2}}^2$$

$$= \left[v - \frac{1}{2}\ln\left|\frac{v+1}{v-1}\right|\right]_{\sqrt{2}}^2 = 2 - \tfrac{1}{2}\ln 3 - \sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2 - \sqrt{2} + \ln\left(\sqrt{2} + 1\right) - \tfrac{1}{2}\ln 3$$

Or: Use Formula 23 in the table of integrals.

13. $y = \cosh x \Rightarrow y' = \sinh x \Rightarrow 1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$.
So $L = \int_0^1 \cosh x\,dx = [\sinh x]_0^1 = \sinh 1 = \frac{1}{2}(e - 1/e)$.

14. $y^2 = 4x,\ x = \frac{1}{4}y^2 \Rightarrow \frac{dx}{dy} = \frac{1}{2}y \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y^2$. So

$$L = \int_0^2 \sqrt{1 + \tfrac{1}{4}y^2}\,dy = \int_0^1 \sqrt{1+u^2}\cdot 2\,du \qquad [u = \tfrac{1}{2}y,\ dy = 2\,du]$$
$$\overset{21}{=} \left[u\sqrt{1+u^2} + \ln\left|u + \sqrt{1+u^2}\right|\right]_0^1 = \sqrt{2} + \ln(1+\sqrt{2})$$

15. $y = e^x \Rightarrow y' = e^x \Rightarrow 1 + (y')^2 = 1 + e^{2x}$. So

$$L = \int_0^1 \sqrt{1+e^{2x}}\,dx = \int_1^e \sqrt{1+u^2}\,\frac{du}{u} \qquad [u = e^x, \text{ so } x = \ln u,\ dx = du/u]$$
$$= \int_1^e \frac{\sqrt{1+u^2}}{u^2}\,u\,du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{v}{v^2-1}\,v\,dv \qquad \left[v = \sqrt{1+u^2}, \text{ so } v^2 = 1+u^2,\ v\,dv = u\,du\right]$$
$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1/2}{v-1} - \frac{1/2}{v+1}\right) dv = \left[v + \frac{1}{2}\ln\frac{v-1}{v+1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}}$$
$$= \sqrt{1+e^2} + \frac{1}{2}\ln\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} - \sqrt{2} - \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}$$
$$= \sqrt{1+e^2} - \sqrt{2} + \ln\left(\sqrt{1+e^2}-1\right) - 1 - \ln\left(\sqrt{2}-1\right)$$

Or: Use Formula 23 for $\int \left(\sqrt{1+u^2}/u\right)du$, or substitute $u = \tan\theta$.

16. $y = \ln\left(\frac{e^x+1}{e^x-1}\right) = \ln(e^x+1) - \ln(e^x-1) \Rightarrow y' = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1} = \frac{-2e^x}{e^{2x}-1} \Rightarrow$

$1 + (y')^2 = 1 + \frac{4e^{2x}}{(e^{2x}-1)^2} = \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2} \Rightarrow \sqrt{1+(y')^2} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{\cosh x}{\sinh x}$.

So $L = \int_a^b \frac{\cosh x}{\sinh x}\,dx = \left[\ln\sinh x\right]_a^b = \ln\sinh b - \ln\sinh a = \ln\left(\frac{\sinh b}{\sinh a}\right) = \ln\left(\frac{e^b - e^{-b}}{e^a - e^{-a}}\right)$.

17. $y = \cos x \Rightarrow dy/dx = -\sin x \Rightarrow 1 + (dy/dx)^2 = 1 + \sin^2 x$. So $L = \int_0^{2\pi} \sqrt{1+\sin^2 x}\,dx$.

18. $y = 2^x \Rightarrow dy/dx = (2^x)\ln 2 \Rightarrow L = \int_0^3 \sqrt{1 + (\ln 2)^2\, 2^{2x}}\,dx$

19. $x = y + y^3 \Rightarrow dx/dy = 1 + 3y^2 \Rightarrow 1 + (dx/dy)^2 = 1 + (1+3y^2)^2 = 9y^4 + 6y^2 + 2$.
So $L = \int_1^4 \sqrt{9y^4 + 6y^2 + 2}\,dy$.

20. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ y = \pm b\sqrt{1 - x^2/a^2} = \pm\frac{b}{a}\sqrt{a^2-x^2}$ [assume $a > 0$].

$y = \frac{b}{a}\sqrt{a^2-x^2} \Rightarrow \frac{dy}{dx} = \frac{-bx}{a\sqrt{a^2-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{b^2x^2}{a^2(a^2-x^2)}$.

So $L = 2\int_{-a}^{a}\left[1 + \frac{b^2x^2}{a^2(a^2-x^2)}\right]^{1/2} dx = \frac{4}{a}\int_0^a \left[\frac{(b^2-a^2)x^2 + a^4}{a^2-x^2}\right]^{1/2} dx$.