

36. $s_{4n-1} = c_0 + c_1x + c_2x^2 + c_3x^3 + c_0x^4 + c_1x^5 + c_2x^6 + c_3x^7 + \dots + c_3x^{4n-1}$
 $= (c_0 + c_1x + c_2x^2 + c_3x^3)(1 + x^4 + x^8 + \dots + x^{4n-4}) \rightarrow \frac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4}$ as $n \rightarrow \infty$

[by (12.2.4) with $r = x^4$] for $|x^4| < 1 \Leftrightarrow |x| < 1$. Also $s_{4n}, s_{4n+1}, s_{4n+2}$ have the same limits (for example, $s_{4n} = s_{4n-1} + c_0x^{4n}$ and $x^{4n} \rightarrow 0$ for $|x| < 1$). So if at least one of $c_0, c_1, c_2,$ and c_3 is nonzero, then the interval of convergence is $(-1, 1)$ and $f(x) = \frac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4}$.

37. We use the Root Test on the series $\sum c_n x^n$. We need $\lim_{n \rightarrow \infty} \sqrt[n]{|c_n x^n|} = |x| \lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} = c|x| < 1$ for convergence, or $|x| < 1/c$, so $R = 1/c$.

38. Suppose $c_n \neq 0$. Applying the Ratio Test to the series $\sum c_n(x - a)^n$, we find that

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}(x - a)^{n+1}}{c_n(x - a)^n} \right| = \lim_{n \rightarrow \infty} \frac{|x - a|}{|c_n/c_{n+1}|} (*) = \frac{|x - a|}{\lim_{n \rightarrow \infty} |c_n/c_{n+1}|}$$

(if $\lim_{n \rightarrow \infty} |c_n/c_{n+1}| \neq 0$), so the series converges when $\frac{|x - a|}{\lim_{n \rightarrow \infty} |c_n/c_{n+1}|} < 1 \Leftrightarrow |x - a| < \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|$. Thus, $R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|$. If $\lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| = 0$ and $|x - a| \neq 0$, then (*) shows that $L = \infty$ and so the series diverges, and hence, $R = 0$. Thus, in all cases, $R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|$.

39. For $2 < x < 3$, $\sum c_n x^n$ diverges and $\sum d_n x^n$ converges. By Exercise 12.2.61, $\sum (c_n + d_n) x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum (c_n + d_n) x^n$ is 2.

40. Since $\sum c_n x^n$ converges whenever $|x| < R$, $\sum c_n x^{2n} = \sum c_n (x^2)^n$ converges whenever $|x^2| < R \Leftrightarrow |x| < \sqrt{R}$, so the second series has radius of convergence \sqrt{R} .

12.9 Representations of Functions as Power Series

- If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 10, then $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.
- If $f(x) = \sum_{n=0}^{\infty} b_n x^n$ converges on $(-2, 2)$, then $\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence—it may happen that the integrated series converges at an endpoint (or both endpoints).
- Our goal is to write the function in the form $\frac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$ with $|x| < 1 \Leftrightarrow |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \frac{3}{1 - x^4}$
 $|x^4| < 1 \Leftrightarrow$
 converges, so the

5. Replacing x with
 $\Leftrightarrow |x|^3 < 1 \Leftrightarrow$

6. $f(x) = \frac{1}{1 + 9x^2}$
 that is, when $|x|$

7. $f(x) = \frac{1}{x - 5}$
 $\left| \frac{x}{5} \right| < 1$; that is,

8. $f(x) = \frac{x}{4x + 1}$
 that is, when $|x|$

9. $f(x) = \frac{x}{9 + x^2}$
 $= \frac{x}{9} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{9^n}$
 $\left| -\left(\frac{x}{3}\right)^2 \right| < 1$

10. $f(x) = \frac{x^2}{a^3 - x^3}$
 $|x^3/a^3| < 1 \Leftrightarrow$

11. $f(x) = \frac{3}{x^2 + x}$
 get $A = -1$. Tak
 $\frac{3}{x^2 + x - 1}$

We represented the second converges

$$4. f(x) = \frac{3}{1-x^4} = 3\left(\frac{1}{1-x^4}\right) = 3(1+x^4+x^8+x^{12}+\dots) = 3\sum_{n=0}^{\infty}(x^4)^n = \sum_{n=0}^{\infty}3x^{4n} \text{ with}$$

$$|x^4| < 1 \Leftrightarrow |x| < 1, \text{ so } R = 1 \text{ and } I = (-1, 1). \text{ [Note that } 3\sum_{n=0}^{\infty}(x^4)^n \text{ converges } \Leftrightarrow \sum_{n=0}^{\infty}(x^4)^n$$

converges, so the appropriate condition (from equation (1)) is $|x^4| < 1$.]

$$5. \text{ Replacing } x \text{ with } x^3 \text{ in (1) gives } f(x) = \frac{1}{1-x^3} = \sum_{n=0}^{\infty}(x^3)^n = \sum_{n=0}^{\infty}x^{3n}. \text{ The series converges when } |x^3| < 1$$

$$\Leftrightarrow |x|^3 < 1 \Leftrightarrow |x| < \sqrt[3]{1} \Leftrightarrow |x| < 1. \text{ Thus, } R = 1 \text{ and } I = (-1, 1).$$

$$6. f(x) = \frac{1}{1+9x^2} = \frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-1)^n 3^{2n} x^{2n}. \text{ The series converges when } |-9x^2| < 1;$$

that is, when $|x| < \frac{1}{3}$, so $I = (-\frac{1}{3}, \frac{1}{3})$.

$$7. f(x) = \frac{1}{x-5} = -\frac{1}{5}\left(\frac{1}{1-x/5}\right) = -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n \text{ or equivalently, } -\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}x^n. \text{ The series converges when}$$

$$\left|\frac{x}{5}\right| < 1; \text{ that is, when } |x| < 5, \text{ so } I = (-5, 5).$$

$$8. f(x) = \frac{x}{4x+1} = x \cdot \frac{1}{1-(-4x)} = x \sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-1)^n 2^{2n} x^{n+1}. \text{ The series converges when } |-4x| < 1;$$

that is, when $|x| < \frac{1}{4}$, so $I = (-\frac{1}{4}, \frac{1}{4})$.

$$9. f(x) = \frac{x}{9+x^2} = \frac{x}{9}\left[\frac{1}{1+(x/3)^2}\right] = \frac{x}{9}\left[\frac{1}{1-\{-(x/3)^2\}}\right] = \frac{x}{9}\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n$$

$$= \frac{x}{9}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{9^n} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{9^{n+1}}. \text{ The geometric series } \sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n \text{ converges when}$$

$$\left|-\left(\frac{x}{3}\right)^2\right| < 1 \Leftrightarrow \frac{|x^2|}{9} < 1 \Leftrightarrow |x|^2 < 9 \Leftrightarrow |x| < 3, \text{ so } R = 3 \text{ and } I = (-3, 3).$$

$$10. f(x) = \frac{x^2}{a^3-x^3} = \frac{x^2}{a^3} \cdot \frac{1}{1-x^3/a^3} = \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x^3}{a^3}\right)^n = \sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}. \text{ The series converges when}$$

$$|x^3/a^3| < 1 \Leftrightarrow |x^3| < |a^3| \Leftrightarrow |x| < |a|, \text{ so } R = |a| \text{ and } I = (-|a|, |a|).$$

$$11. f(x) = \frac{3}{x^2+x-2} = \frac{3}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \Rightarrow 3 = A(x-1) + B(x+2). \text{ Taking } x = -2, \text{ we}$$

get $A = -1$. Taking $x = 1$, we get $B = 1$. Thus,

$$\frac{3}{x^2+x-2} = \frac{1}{x-1} - \frac{1}{x+2} = -\frac{1}{1-x} - \frac{1}{2}\frac{1}{1+x/2} = -\sum_{n=0}^{\infty}x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[-1 - \frac{1}{2}\left(-\frac{1}{2}\right)^n\right]x^n = \sum_{n=0}^{\infty}\left[-1 + \left(-\frac{1}{2}\right)^{n+1}\right]x^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{2^{n+1}} - 1\right]x^n$$

We represented the given function as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-2, 2)$. Thus, the sum converges for $x \in (-1, 1) = I$.

$$12. f(x) = \frac{7x-1}{3x^2+2x-1} = \frac{7x-1}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1} = \frac{1}{3x-1} + \frac{2}{x+1} = 2 \cdot \frac{1}{1-(-x)} - \frac{1}{1-3x}$$

$$= 2 \sum_{n=0}^{\infty} (-x)^n - \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} [2(-1)^n - 3^n] x^n$$

The series $\sum (-x)^n$ converges for $x \in (-1, 1)$ and the series $\sum (3x)^n$ converges for $x \in (-\frac{1}{3}, \frac{1}{3})$, so their common converges for $x \in (-\frac{1}{3}, \frac{1}{3}) = I$.

$$13. (a) f(x) = \frac{1}{(1+x)^2} = \frac{d}{dx} \left(\frac{-1}{1+x} \right) = -\frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] \text{ [from Exercise 31]}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} \text{ [from Theorem 2(i)]} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \text{ with } R = 1.$$

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$]

$$(b) f(x) = \frac{1}{(1+x)^3} = -\frac{1}{2} \frac{d}{dx} \left[\frac{1}{(1+x)^2} \right] = -\frac{1}{2} \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n (n+1) x^n \right] \text{ [from part (a)]}$$

$$= -\frac{1}{2} \sum_{n=1}^{\infty} (-1)^n (n+1) n x^{n-1} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n \text{ with } R = 1.$$

$$(c) f(x) = \frac{x^2}{(1+x)^3} = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n \text{ [from part (b)]}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^{n+2}. \text{ To write the power series with } x^n \text{ rather than } x^{n+2},$$

we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable

by 2. This gives us $\frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n)(n-1) x^n$.

$$14. (a) \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n \text{ [geometric series with } R = 1], \text{ so}$$

$$f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \text{ [} C = 0 \text{ since } f(0) = \ln 1 = 0], \text{ with } R = 1$$

$$(b) f(x) = \ln(1+x) = \int \left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \right] dx \text{ [by part (a)]} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n(n+1)} = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n-1}$$

with $R = 1$.

$$(c) f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x^2)^n}{n} \text{ [by part (a)]} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} \text{ with } R = 1.$$

$$\therefore f(x) = \ln(5-x) = -\int \frac{dx}{5-x} = -\frac{1}{5} \int \frac{dx}{1-x/5}$$

$$= -\frac{1}{5} \int \left[\sum_{n=0}^{\infty} \left(\frac{x}{5} \right)^n \right] dx = C - \frac{1}{5} \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \Leftrightarrow |x| < 5$, so $R = 5$.

16. We know

$$f(x) = \frac{1}{2-x}$$

with $R = 2$

$$17. \frac{1}{2-x} = \frac{1}{(x-2)^2}$$

$$f(x) = \frac{1}{(x-2)^2}$$

$$f(x) = \frac{1}{(x-2)^2}$$

$$I = (-2, 2)$$

18. From Exan

$$f(x) = \arctan(x)$$

so $R = 3$.

19. $f(x) = \ln(3-x)$

$$= C - \ln(3-x)$$

$$= \ln 3$$

The terms of

As n increases,

16. We know that $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$. Differentiating, we get $\frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1} = \sum_{n=0}^{\infty} 2^{n+1} (n+1) x^n$, so

$$f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \cdot \frac{2}{(1-2x)^2} = \frac{x^2}{2} \sum_{n=0}^{\infty} 2^{n+1} (n+1) x^n = \sum_{n=0}^{\infty} 2^n (n+1) x^{n+2} \text{ or } \sum_{n=2}^{\infty} 2^{n-2} (n-1) x^n,$$

with $R = \frac{1}{2}$.

17. $\frac{1}{2-x} = \frac{1}{2(1-x/2)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n$ for $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$. Now

$$\frac{1}{(x-2)^2} = \frac{d}{dx} \left(\frac{1}{2-x} \right) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n \right) = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} x^{n-1} = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n. \text{ So}$$

$$f(x) = \frac{x^3}{(x-2)^2} = x^3 \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^{n+3} \text{ or } \sum_{n=3}^{\infty} \frac{n-2}{2^{n-1}} x^n \text{ for } |x| < 2. \text{ Thus, } R = 2 \text{ and}$$

$$I = (-2, 2).$$

18. From Example 7, $g(x) = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. Thus,

$$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} x^{2n+1} \text{ for } \left|\frac{x}{3}\right| < 1 \Leftrightarrow |x| < 3,$$

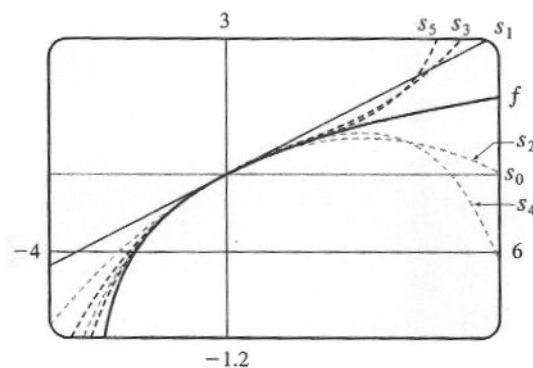
so $R = 3$.

19. $f(x) = \ln(3+x) = \int \frac{dx}{3+x} = \frac{1}{3} \int \frac{dx}{1+x/3} = \frac{1}{3} \int \frac{dx}{1-(-x/3)} = \frac{1}{3} \int \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n dx$

$$= C + \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)3^n} x^{n+1} = \ln 3 + \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^{n-1}} x^n \quad [C = f(0) = \ln 3]$$

$$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^n} x^n. \text{ The series converges when } |-x/3| < 1 \Leftrightarrow |x| < 3, \text{ so } R = 3.$$

The terms of the series are $a_0 = \ln 3$, $a_1 = \frac{x}{3}$, $a_2 = -\frac{x^2}{18}$, $a_3 = \frac{x^3}{81}$, $a_4 = -\frac{x^4}{324}$, $a_5 = \frac{x^5}{1215}$, \dots

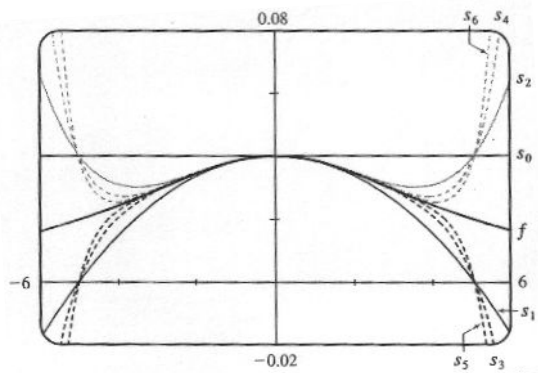


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-3, 3)$.

$$20. f(x) = \frac{1}{x^2 + 25} = \frac{1}{25} \left(\frac{1}{1 + x^2/25} \right) = \frac{1}{25} \left(\frac{1}{1 - (-x^2/25)} \right) = \frac{1}{25} \sum_{n=0}^{\infty} \left(-\frac{x^2}{25} \right)^n = \frac{1}{25} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^2}{25} \right)^n$$

The series converges when $|-x^2/25| < 1 \Leftrightarrow x^2 < 25 \Leftrightarrow |x| < 5$, so $R = 5$. The terms of the series

$$a_0 = \frac{1}{25}, a_1 = -\frac{x^2}{625}, a_2 = \frac{x^4}{15,625}, \dots$$

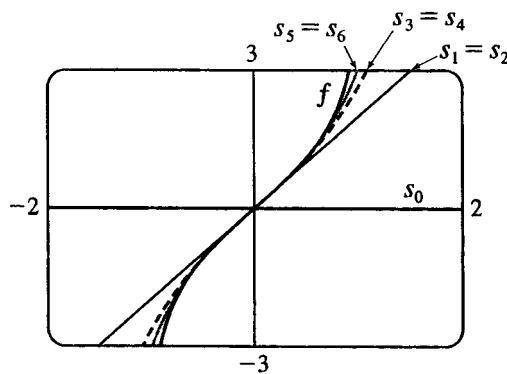


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-5, 5)$.

$$\begin{aligned} 21. f(x) &= \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x} \\ &= \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x} = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n \right] dx \\ &= \int [(1-x+x^2-x^3+x^4-\dots) + (1+x+x^2+x^3+x^4+\dots)] dx \\ &= \int (2+2x^2+2x^4+\dots) dx = \int \sum_{n=0}^{\infty} 2x^{2n} dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} \end{aligned}$$

But $f(0) = \ln \frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then

$$f(x) = \pm 2 \sum_{n=0}^{\infty} \frac{1}{2n+1}, \text{ which both diverge by the Limit Comparison Test with } b_n = \frac{1}{n}.$$

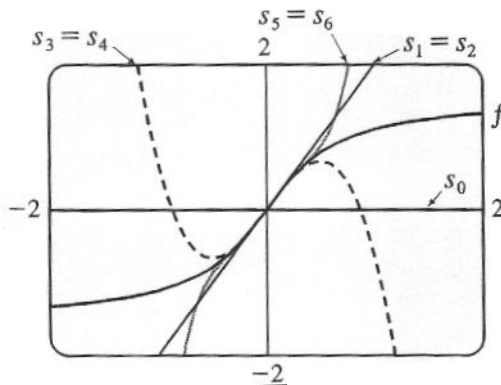


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

$$\begin{aligned}
 22. f(x) &= \tan^{-1}(2x) = 2 \int \frac{dx}{1+4x^2} = 2 \int \sum_{n=0}^{\infty} (-1)^n (4x^2)^n dx = 2 \int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} dx \\
 &= C + 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \quad [f(0) = \tan^{-1} 0 = 0, \text{ so } C = 0].
 \end{aligned}$$

The series converges when $|4x^2| < 1 \Leftrightarrow |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm \frac{1}{2}$, then $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and

$$f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}, \text{ respectively. Both series converge by the Alternating Series Test.}$$



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $[-\frac{1}{2}, \frac{1}{2}]$.

$$\begin{aligned}
 23. \frac{t}{1-t^8} &= t \cdot \frac{1}{1-t^8} = t \sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \Rightarrow \int \frac{t}{1-t^8} dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}. \text{ The series for } \frac{1}{1-t^8} \\
 &\text{converges when } |t^8| < 1 \Leftrightarrow |t| < 1, \text{ so } R = 1 \text{ for that series and also the series for } t/(1-t^8). \text{ By Theorem} \\
 &2, \text{ the series for } \int \frac{t}{1-t^8} dt \text{ also has } R = 1.
 \end{aligned}$$

$$\begin{aligned}
 24. \text{ By Example 6, } \ln(1-t) &= -\sum_{n=1}^{\infty} \frac{t^n}{n} \text{ for } |t| < 1, \text{ so } \frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \frac{t^{n-1}}{n} \text{ and} \\
 \int \frac{\ln(1-t)}{t} dt &= C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}. \text{ By Theorem 2, } R = 1.
 \end{aligned}$$

$$25. \text{ By Example 7, } \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ with } R = 1, \text{ so}$$

$$x - \tan^{-1} x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

$$\text{and } \frac{x - \tan^{-1} x}{x^3} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{2n+1}, \text{ so}$$

$$\int \frac{x - \tan^{-1} x}{x^3} dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{4n^2-1}. \text{ By Theorem 2, } R = 1.$$