38. (a) Following the hint, we get that $|a_n| < r^n$ for $n \geq N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges ($0 < r < 1$), the series $\sum_{n=N}^{\infty} |a_n|$ converges as well by the Comparison Test, and hence so does $\sum_{n=1}^{\infty} |a_n|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then there is an integer N such that $\sqrt[n]{|a_n|} > 1$ for all $n \geq N$, so $|a_n| > 1$ for $n \geq N$. Thus, $\lim\limits_{n\to\infty} a_n \neq 0$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

39. (a) Since $\sum a_n$ is absolutely convergent, and since $|a_n^+| \leq |a_n|$ and $|a_n^-| \leq |a_n|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Comparison Test that both $\sum a_n^+$ and $\sum a_n^-$ must be absolutely convergent. (Or use Theorem 12.2.8.)

(b) We will show by contradiction that both $\sum a_n^+$ and $\sum a_n^-$ must diverge. For suppose that $\sum a_n^+$ converged. Then so would $\sum\left(a_n^+ - \frac{1}{2}a_n\right)$ by Theorem 12.2.8. But $\sum\left(a_n^+ - \frac{1}{2}a_n\right) = \sum\left[\frac{1}{2}\left(a_n + |a_n|\right) - \frac{1}{2}a_n\right] = \frac{1}{2}\sum |a_n|$, which diverges because $\sum a_n$ is only conditionally convergent. Hence, $\sum a_n^+$ can't converge. Similarly, neither can $\sum a_n^-$.

40. Let $\sum b_n$ be the rearranged series constructed in the hint. [This series can be constructed by virtue of the result of Exercise 39(b).] This series will have partial sums s_n that oscillate in value back and forth across r. Since $\lim\limits_{n\to\infty} a_n = 0$ (by Theorem 12.2.6), and since the size of the oscillations $|s_n - r|$ is always less than $|a_n|$ because of the way $\sum b_n$ was constructed, we have that $\sum b_n = \lim\limits_{n\to\infty} s_n = r$.

12.7 Strategy for Testing Series

1. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^2-1}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{1-1/n^2}{1+1/n} = 1 \neq 0$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{n^2-1}{n^2+1}$ diverges by the Test for Divergence.

2. If $a_n = \dfrac{n-1}{n^2+n}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^2-n}{n^2+n} = \lim\limits_{n\to\infty} \dfrac{1-1/n}{1+1/n} = 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{n-1}{n^2+n}$ diverges by the Limit Comparison Test with the harmonic series.

3. $\dfrac{1}{n^2+n} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+n}$ converges by the Comparison Test with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, a p-series that converges because $p = 2 > 1$.

4. Let $b_n = \dfrac{n-1}{n^2+n}$. Then $b_1 = 0$, and $b_2 = b_3 = \frac{1}{6}$, but $b_n > b_{n+1}$ for $n \geq 3$ since

$$\left(\frac{x-1}{x^2+x}\right)' = \frac{(x^2+x)-(x-1)(2x+1)}{(x^2+x)^2} = \frac{-x^2+2x+1}{(x^2+x)^2} = \frac{2-(x-1)^2}{(x^2+x)^2} < 0 \text{ for } x \geq 3. \text{ Thus,}$$

$\{b_n \mid n \geq 3\}$ is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=3}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ converges by the Alternating Series Test.

Hence, the full series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ also converges.

5. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \dfrac{2^{3n}}{(-3)^{n+1}}\right| = \lim\limits_{n\to\infty} \left|\dfrac{-3 \cdot 2^{3n}}{2^{3n} \cdot 2^3}\right| = \lim\limits_{n\to\infty} \dfrac{3}{2^3} = \dfrac{3}{8} < 1$, so the series

$\sum\limits_{n=1}^{\infty} \dfrac{(-3)^{n+1}}{2^{3n}}$ is absolutely convergent by the Ratio Test.

6. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{3n}{1+8n} = \lim\limits_{n\to\infty} \dfrac{3}{1/n+8} = \dfrac{3}{8} < 1$, so $\sum\limits_{n=1}^{\infty} \left(\dfrac{3n}{1+8n}\right)^n$ converges by the Root Test.

7. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2, \infty)$, so we can apply the Integral Test.

Since $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\,dx \quad \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\displaystyle\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral

diverges, the given series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$ diverges.

8. $\sum\limits_{k=1}^{\infty} \dfrac{2^k k!}{(k+2)!} = \sum\limits_{k=1}^{\infty} \dfrac{2^k}{(k+1)(k+2)}$. Using the Ratio Test, we get

$\lim\limits_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \left|\dfrac{2^{k+1}}{(k+2)(k+3)} \cdot \dfrac{(k+1)(k+2)}{2^k}\right| = \lim\limits_{k\to\infty} \left(2 \cdot \dfrac{k+1}{k+3}\right) = 2 > 1$, so the series diverges.

Or: Use the Test for Divergence.

9. $\sum\limits_{k=1}^{\infty} k^2 e^{-k} = \sum\limits_{k=1}^{\infty} \dfrac{k^2}{e^k}$. Using the Ratio Test, we get

$\lim\limits_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \left|\dfrac{(k+1)^2}{e^{k+1}} \cdot \dfrac{e^k}{k^2}\right| = \lim\limits_{k\to\infty} \left[\left(\dfrac{k+1}{k}\right)^2 \cdot \dfrac{1}{e}\right] = 1^2 \cdot \dfrac{1}{e} = \dfrac{1}{e} < 1$, so the series converges.

10. Let $f(x) = x^2 e^{-x^3}$ Then f is continuous and positive on $[1, \infty)$, and $f'(x) = \dfrac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is decreasing on $[1, \infty)$ as well, and we can apply the Integral Test. $\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim\limits_{t\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_1^t = \frac{1}{3e}$, so the integral converges, and hence, the series converges.

11. $b_n = \dfrac{1}{n \ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the given series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{n \ln n}$ converges by the Alternating Series Test.

12. Let $b_n = \dfrac{n}{n^2+25}$. Then $b_n > 0$, $\lim\limits_{n\to\infty} b_n = 0$, and

$b_n - b_{n+1} = \dfrac{n}{n^2+25} - \dfrac{n+1}{n^2+2n+26} = \dfrac{n^2+n-25}{(n^2+25)(n^2+2n+26)}$, which is positive for $n \geq 5$, so the sequence $\{b_n\}$ decreases from $n = 5$ on. Hence, the given series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n}{n^2+25}$ converges by the Alternating Series Test.

13. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \dfrac{n!}{3^n n^2}\right| = \lim\limits_{n\to\infty} \left[\dfrac{3(n+1)^2}{(n+1)n^2}\right] = 3 \lim\limits_{n\to\infty} \dfrac{n+1}{n^2} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{3^n n^2}{n!}$ converges by the Ratio Test.

14. The series $\sum_{n=1}^{\infty} \sin n$ diverges by the Test for Divergence since $\lim\limits_{n\to\infty} \sin n$ does not exist.

15. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)!}{2\cdot 5\cdot 8\cdot \cdots \cdot (3n+2)[3(n+1)+2]} \cdot \dfrac{2\cdot 5\cdot 8\cdot \cdots \cdot (3n+2)}{n!}\right|$

$= \lim\limits_{n\to\infty} \dfrac{n+1}{3n+5} = \dfrac{1}{3} < 1$

so the series $\sum\limits_{n=0}^{\infty} \dfrac{n!}{2\cdot 5\cdot 8\cdot \cdots \cdot (3n+2)}$ converges by the Ratio Test.

16. Using the Limit Comparison Test with $a_n = \dfrac{n^2+1}{n^3+1}$ and $b_n = \dfrac{1}{n}$, we have

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \left(\dfrac{n^2+1}{n^3+1} \cdot \dfrac{n}{1}\right) = \lim\limits_{n\to\infty} \dfrac{n^3+n}{n^3+1} = \lim\limits_{n\to\infty} \dfrac{1+1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

17. $\lim\limits_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\lim\limits_{n\to\infty} (-1)^n\, 2^{1/n}$ does not exist and the series $\sum\limits_{n=1}^{\infty} (-1)^n 2^{1/n}$ diverges by the Test for Divergence.

18. $b_n = \dfrac{1}{\sqrt{n}-1}$ for $n \geq 2$. $\{b_n\}$ is a decreasing sequence of positive numbers and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the Alternating Series Test.

19. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$. Then $f'(x) = \dfrac{2-\ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\dfrac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{1/n}{1/(2\sqrt{n})} = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n}} = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

20. $\lim_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\dfrac{k+6}{5^{k+1}}\cdot\dfrac{5^k}{k+5}\right| = \dfrac{1}{5}\lim_{k\to\infty}\dfrac{k+6}{k+5} = \dfrac{1}{5} < 1$, so the series $\sum\limits_{k=1}^{\infty}\dfrac{k+5}{5^k}$ converges by the Ratio Test.

21. $\sum\limits_{n=1}^{\infty}\dfrac{(-2)^{2n}}{n^n} = \sum\limits_{n=1}^{\infty}\left(\dfrac{4}{n}\right)^n$. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\dfrac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.

22. $\dfrac{\sqrt{n^2-1}}{n^3+2n^2+5} < \dfrac{n}{n^3+2n^2+5} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for $n \geq 1$, so $\sum\limits_{n=1}^{\infty}\dfrac{\sqrt{n^2-1}}{n^3+2n^2+5}$ converges by the Comparison Test with the convergent p-series $\sum_{n=1}^{\infty} 1/n^2$ $(p = 2 > 1)$.

23. Using the Limit Comparison Test with $a_n = \tan\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\tan(1/n)}{1/n} = \lim_{x\to\infty}\frac{\tan(1/x)}{1/x} \overset{H}{=} \lim_{x\to\infty}\frac{\sec^2(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty}\sec^2(1/x) = 1^2 = 1 > 0.$$

Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

24. $\dfrac{|\cos(n/2)|}{n^2+4n} < \dfrac{1}{n^2+4n} < \dfrac{1}{n^2}$ and since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ converges $(p = 2 > 1)$, $\sum\limits_{n=1}^{\infty}\dfrac{\cos(n/2)}{n^2+4n}$ converges absolutely by the Comparison Test.

25. Use the Ratio Test. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)!}{e^{(n+1)^2}}\cdot\dfrac{e^{n^2}}{n!}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)n!\cdot e^{n^2}}{e^{n^2+2n+1}n!} = \lim\limits_{n\to\infty}\dfrac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum\limits_{n=1}^{\infty}\dfrac{n!}{e^{n^2}}$ converges.

26. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\left(\dfrac{n^2+2n+2}{5^{n+1}}\cdot\dfrac{5^n}{n^2+1}\right) = \lim\limits_{n\to\infty}\left(\dfrac{1+2/n+2/n^2}{1+1/n^2}\cdot\dfrac{1}{5}\right) = \dfrac{1}{5} < 1$, so $\sum\limits_{n=1}^{\infty}\dfrac{n^2+1}{5^n}$ converges by the Ratio Test.

27. $\displaystyle\int_2^{\infty}\frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ (using integration by parts) $\overset{H}{=} 1$. So $\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^2}$ converges by the Integral Test, and since $\dfrac{k\ln k}{(k+1)^3} < \dfrac{k\ln k}{k^3} = \dfrac{\ln k}{k^2}$, the given series $\sum\limits_{k=1}^{\infty}\dfrac{k\ln k}{(k+1)^3}$ converges by the Comparison Test.

28. Since $\left\{\dfrac{1}{n}\right\}$ is a decreasing sequence, $e^{1/n} \leq e^{1/1} = e$ for all $n \geq 1$, and $\sum\limits_{n=1}^{\infty}\dfrac{e}{n^2}$ converges $(p = 2 > 1)$, so $\sum\limits_{n=1}^{\infty}\dfrac{e^{1/n}}{n^2}$ converges by the Comparison Test. (Or use the Integral Test.)

29. $0 < \dfrac{\tan^{-1} n}{n^{3/2}} < \dfrac{\pi/2}{n^{3/2}}$. $\sum\limits_{n=1}^{\infty}\dfrac{\pi/2}{n^{3/2}} = \dfrac{\pi}{2}\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{3/2}}$ which is a convergent p-series $(p = \frac{3}{2} > 1)$, so $\sum\limits_{n=1}^{\infty}\dfrac{\tan^{-1} n}{n^{3/2}}$ converges by the Comparison Test.

30. Let $f(x) = \dfrac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1, \infty)$, and since $f'(x) = \dfrac{5-x}{2\sqrt{x}\,(x+5)^2} < 0$ for $x > 5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test.

$\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n+5} = \lim_{n\to\infty} \frac{1}{n^{1/2} + 5n^{-1/2}} = 0$, so the series $\displaystyle\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$ converges.

31. $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{5^k}{3^k + 4^k} = [\text{divide by } 4^k] \lim_{k\to\infty} \frac{(5/4)^k}{(3/4)^k + 1} = \infty$ since $\displaystyle\lim_{k\to\infty} \left(\frac{3}{4}\right)^k = 0$ and $\displaystyle\lim_{k\to\infty} \left(\frac{5}{4}\right)^k = \infty$.

Thus, $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

32. $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} = 0$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ converges by the Root Test.

33. Let $a_n = \dfrac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \dfrac{1}{n\sqrt{n}}$. Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 > 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges by limit comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $(p = 3/2 > 1)$.

34. $0 \le n\cos^2 n \le n$, so $\dfrac{1}{n + n\cos^2 n} \ge \dfrac{1}{n+n} = \dfrac{1}{2n}$. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$ diverges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$, which is a constant multiple of the (divergent) harmonic series.

35. $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2/n} = \lim_{n\to\infty} \frac{1}{[(n+1)/n]^n} = \frac{1}{\lim\limits_{n\to\infty} (1 + 1/n)^n} = \frac{1}{e} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

36. Note that $(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n}$ and $\ln \ln n \to \infty$ as $n \to \infty$, so $\ln \ln n > 2$ for sufficiently large n. For these n we have $(\ln n)^{\ln n} > n^2$, so $\dfrac{1}{(\ln n)^{\ln n}} < \dfrac{1}{n^2}$. Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges $(p = 2 > 1)$, so does $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

37. $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(2^{1/n} - 1\right) = 1 - 1 = 0 < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ converges by the Root Test.