12.6 Absolute Convergence and the Ratio and Root Tests

1. (a) Since $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

2. The series $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{2^n}$ has positive terms and $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\left[\dfrac{(n+1)^2}{2^{n+1}}\cdot\dfrac{2^n}{n^2}\right] = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^2\cdot\dfrac{1}{2} = \dfrac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

3. $\sum\limits_{n=0}^{\infty}\dfrac{(-10)^n}{n!}$. Using the Ratio Test, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-10)^{n+1}}{(n+1)!}\cdot\dfrac{n!}{(-10)^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{-10}{n+1}\right| = 0 < 1$, so the series is absolutely convergent.

4. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim\limits_{n\to\infty}\dfrac{2^n}{n^4} = \infty$, so $\lim\limits_{n\to\infty}(-1)^{n-1}\dfrac{2^n}{n^4}$ does not exist.

5. $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by the Alternating Series Test, but $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt[4]{n}}$ is a divergent p-series $\left(p = \frac{1}{4} \le 1\right)$, so the given series is conditionally convergent.

6. $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^4}$ is a convergent p-series $(p = 4 > 1)$, so $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^4}$ is absolutely convergent.

7. $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}\dfrac{n}{5+n} = \lim\limits_{n\to\infty}\dfrac{1}{5/n+1} = 1$, so $\lim\limits_{n\to\infty}a_n \ne 0$. Thus, the given series is divergent by the Test for Divergence.

8. $\sum\limits_{n=1}^{\infty}\dfrac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series:

$\lim\limits_{n\to\infty}\dfrac{n/(n^2+1)}{1/n} = \lim\limits_{n\to\infty}\dfrac{n^2}{n^2+1} = 1$. But $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+1}$ converges by the Alternating Series Test:

$\left\{\dfrac{n}{n^2+1}\right\}$ has positive terms, is decreasing since $\left(\dfrac{x}{x^2+1}\right)' = \dfrac{1-x^2}{(x^2+1)^2} \le 0$ for $x \ge 1$, and

$\lim\limits_{n\to\infty}\dfrac{n}{n^2+1} = 0$. Thus, $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+1}$ is conditionally convergent.

9. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{1/(2n+2)!}{1/(2n)!} = \lim\limits_{n\to\infty}\dfrac{(2n)!}{(2n+2)!} = \lim\limits_{n\to\infty}\dfrac{(2n)!}{(2n+2)(2n+1)(2n)!}$

$= \lim\limits_{n\to\infty}\dfrac{1}{(2n+2)(2n+1)} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{(2n)!}$ is absolutely convergent by the Ratio Test. Of course, absolute convergence is the same as convergence for this series, since all of its terms are positive.

10. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)!/e^{n+1}}{n!/e^n}\right| = \dfrac{1}{e}\lim\limits_{n\to\infty}(n+1) = \infty$, so the series $\sum\limits_{n=1}^{\infty}e^{-n}n!$ diverges by the Ratio Test.

11. Since $0 \le \dfrac{e^{1/n}}{n^3} \le \dfrac{e}{n^3} = e\left(\dfrac{1}{n^3}\right)$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series $(p = 3 > 1)$, $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$ converges, and so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$ is absolutely convergent.

12. $\left|\dfrac{\sin 4n}{4^n}\right| \le \dfrac{1}{4^n}$, so $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$ $\left(|r| = \frac{1}{4} < 1\right)$. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is absolutely convergent.

13. $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)\,3^{n+1}}{4^n} \cdot \frac{4^{n-1}}{n \cdot 3^n}\right] = \lim_{n\to\infty} \left(\frac{3}{4} \cdot \frac{n+1}{n}\right) = \frac{3}{4} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$ is absolutely convergent by the Ratio Test.

14. $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)^2\, 2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 2^n}\right] = \lim_{n\to\infty} \left[\left(1 + \frac{1}{n}\right)^2 \cdot \frac{2}{n+1}\right] = 0$, so the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ is absolutely convergent by the Ratio Test.

15. $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{10^{n+1}}{(n+2)\,4^{2n+3}} \cdot \frac{(n+1)\,4^{2n+1}}{10^n}\right] = \lim_{n\to\infty} \left(\frac{10}{4^2} \cdot \frac{n+1}{n+2}\right) = \frac{5}{8} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

16. $n^{2/3} - 2 > 0$ for $n \ge 3$, so $\dfrac{3 - \cos n}{n^{2/3} - 2} > \dfrac{1}{n^{2/3} - 2} > \dfrac{1}{n^{2/3}}$ for $n \ge 3$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges $\left(p = \frac{2}{3} \le 1\right)$, so does $\displaystyle\sum_{n=1}^{\infty} \frac{3 - \cos n}{n^{2/3} - 2}$ by the Comparison Test.

17. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$ and $\left\{\dfrac{1}{\ln n}\right\}$ is decreasing. Now $\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$, and since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent (partial) harmonic series, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the Comparison Test. Thus, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent.

18. $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges absolutely by the Ratio Test.

19. $\dfrac{|\cos(n\pi/3)|}{n!} \le \dfrac{1}{n!}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ converges (use the Ratio Test or the result of Exercise 12.4.29), so the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ converges absolutely by the Comparison Test.

20. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \dfrac{1}{\ln n} = 0 < 1$, so the series $\sum_{n=2}^{\infty} \dfrac{(-1)^n}{(\ln n)^n}$ converges absolutely by the Root Test.

21. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\dfrac{n^n}{3^{1+3n}}\right)^{1/n} = \lim_{n\to\infty} \dfrac{n}{\sqrt[n]{3}\cdot 3^3} = \infty$, so the series $\sum_{n=1}^{\infty} \dfrac{n^n}{3^{1+3n}}$ is divergent by the Root Test.

Or: $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\dfrac{(n+1)^{n+1}}{3^{4+3n}} \cdot \dfrac{3^{1+3n}}{n^n}\right] = \lim_{n\to\infty} \left[\dfrac{1}{3^3} \cdot \left(\dfrac{n+1}{n}\right)^n (n+1)\right]$

$= \dfrac{1}{27} \lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n \lim_{n\to\infty} (n+1) = \frac{1}{27} e \lim_{n\to\infty} (n+1) = \infty$,

so the series is divergent by the Ratio Test.

22. Since $\left\{\dfrac{1}{n\ln n}\right\}$ is decreasing and $\lim_{n\to\infty} \dfrac{1}{n\ln n} = 0$, the series $\sum_{n=2}^{\infty} \dfrac{(-1)^n}{n\ln n}$ converges by the Alternating Series Test. Since $\sum_{n=2}^{\infty} \dfrac{1}{n\ln n}$ diverges by the Integral Test (Exercise 12.3.21), the series $\sum_{n=2}^{\infty} \dfrac{(-1)^n}{n\ln n}$ is conditionally convergent.

23. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \dfrac{n^2+1}{2n^2+1} = \lim_{n\to\infty} \dfrac{1+1/n^2}{2+1/n^2} = \dfrac{1}{2} < 1$, so the series $\sum_{n=1}^{\infty} \left(\dfrac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

24. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \dfrac{1}{\arctan n} = \dfrac{1}{\pi/2} = \dfrac{2}{\pi} < 1$, so the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{(\arctan n)^n}$ is absolutely convergent by the Root Test.

25. Use the Ratio Test with the series $1 - \dfrac{1\cdot 3}{3!} + \dfrac{1\cdot 3\cdot 5}{5!} - \dfrac{1\cdot 3\cdot 5\cdot 7}{7!} + \cdots + (-1)^{n-1} \dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n-1)!} + \cdots$

$= \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n-1)!}$.

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)[2(n+1)-1]}{[2(n+1)-1]!} \cdot \frac{(2n-1)!}{(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}\right| \\ &= \lim_{n\to\infty} \left|\frac{(-1)(2n+1)(2n-1)!}{(2n+1)(2n)(2n-1)!}\right| \\ &= \lim_{n\to\infty} \frac{1}{2n} = 0 < 1,\end{aligned}$$

so the given series is absolutely convergent and therefore convergent.

26. Use the Ratio Test with the series $\dfrac{2}{5} + \dfrac{2\cdot 6}{5\cdot 8} + \dfrac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11} + \dfrac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14} + \cdots = \sum_{n=1}^{\infty} \dfrac{2\cdot 6\cdot 10\cdot 14\cdot \cdots \cdot (4n-2)}{5\cdot 8\cdot 11\cdot 14\cdot \cdots \cdot (3n+2)}$.

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{2\cdot 6\cdot 10\cdot \cdots \cdot (4n-2)[4(n+1)-2]}{5\cdot 8\cdot 11\cdot \cdots \cdot (3n+2)[3(n+1)+2]} \cdot \frac{5\cdot 8\cdot 11\cdot \cdots \cdot (3n+2)}{2\cdot 6\cdot 10\cdot \cdots \cdot (4n-2)}\right| \\ &= \lim_{n\to\infty} \frac{4n+2}{3n+5} = \frac{4}{3} > 1,\end{aligned}$$

so the given series is divergent.