## 12.5 Alternating Series

**1.** (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges if $0 < b_{n+1} \leq b_n$ for all $n$ and $\lim\limits_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum $s_n$ as an approximation to the total sum $s$ is the remainder $R_n = s - s_n$ and the size of the error is smaller than $b_{n+1}$; that is, $|R_n| \leq b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

**2.** $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots = \sum\limits_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$. Here $a_n = (-1)^n\frac{n}{n+2}$. Since $\lim\limits_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

**3.** $\frac{4}{7}-\frac{4}{8}+\frac{4}{9}-\frac{4}{10}+\frac{4}{11}-\cdots = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{4}{n+6}$. Now $b_n = \frac{4}{n+6} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

**4.** $\sum\limits_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$. $b_n = \frac{1}{\ln n}$ is positive and $\{b_n\}$ is decreasing; $\lim\limits_{n\to\infty}\frac{1}{\ln n} = 0$, so the series converges by the Alternating Series Test.

**5.** $b_n = \frac{1}{\sqrt{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

**6.** $b_n = \frac{1}{3n-1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

**7.** $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1} = \sum\limits_{n=1}^{\infty}(-1)^n b_n$. Now $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\frac{3-1/n}{2+1/n} = \frac{3}{2} \neq 0$. Since $\lim\limits_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

**8.** $b_n = \frac{2n}{4n^2+1} > 0$, $\{b_n\}$ is decreasing [since

$$b_n - b_{n+1} = \frac{2n}{4n^2+1} - \frac{2n+2}{4n^2+8n+5} = \frac{8n^2+8n-2}{(4n^2+1)(4n^2+8n+5)} > 0 \text{ for } n \geq 1\text{], and}$$

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\frac{2/n}{4+1/n^2} = 0$, so the series $\sum\limits_{n=1}^{\infty}(-1)^n\frac{2n}{4n^2+1}$ converges by the Alternating Series Test.

Alternatively, to show that $\{b_n\}$ is decreasing, we could verify that $\frac{d}{dx}\left(\frac{2x}{4x^2+1}\right) < 0$ for $x \geq 1$.

**9.** $b_n = \frac{1}{4n^2+1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2+1}$ converges by the Alternating Series Test.

**10.** $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty}$ (

(in fact the lim

**11.** $b_n = \frac{n^2}{n^3+4}$

$\left(\frac{x^2}{x^3+4}\right)' =$

$\lim\limits_{n\to\infty} b_n =$ lin

**12.** $b_n = \frac{e^{1/n}}{n} >$

$\left(\frac{e^{1/x}}{x}\right)' =$

$\lim\limits_{n\to\infty} e^{1/n} = 1$

**13.** $\sum\limits_{n=2}^{\infty}(-1)^n\frac{n}{\ln n}$

**14.** $\sum\limits_{n=1}^{\infty}(-1)^{n-1}($

then $f'(x) =$

$\lim\limits_{n\to\infty} b_n =$ lin

**15.** $\sum\limits_{n=1}^{\infty}\frac{\cos n\pi}{n^{3/4}} =$

the Alternating

**16.** $\sin\left(\frac{n\pi}{2}\right) =$ (

decreasing, an

**17.** $\sum\limits_{n=1}^{\infty}(-1)^n \sin$

converges by t

**18.** $\sum\limits_{n=1}^{\infty}(-1)^n \cos$

the Test for Di

**19.** $\frac{n^n}{n!} = \frac{n \cdot n \cdot}{1 \cdot 2 \cdot}$

the Test for Di

**10.** $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2+1/\sqrt{n}} = \frac{1}{2} \neq 0$. Since $\lim_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

**11.** $b_n = \frac{n^2}{n^3+4} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since

$\left(\frac{x^2}{x^3+4}\right)' = \frac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2} = \frac{x(2x^3+8-3x^3)}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2} < 0$ for $x > 2$. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1/n}{1+4/n^3} = 0$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ converges by the Alternating Series Test.

**12.** $b_n = \frac{e^{1/n}}{n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing since

$\left(\frac{e^{1/x}}{x}\right)' = \frac{x \cdot e^{1/x}(-1/x^2) - e^{1/x} \cdot 1}{x^2} = \frac{-e^{1/x}(1+x)}{x^3} < 0$ for $x > 0$. Also, $\lim_{n\to\infty} b_n = 0$ since

$\lim_{n\to\infty} e^{1/n} = 1$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n}$ converges by the Alternating Series Test.

**13.** $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$. $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{1/x} = \infty$, so the series diverges by the Test for Divergence.

**14.** $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n}\right) = 0 + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n}\right)$. $b_n = \frac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \frac{\ln x}{x}$,

then $f'(x) = \frac{1 - \ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

**15.** $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

**16.** $\sin\left(\frac{n\pi}{2}\right) = 0$ if $n$ is even and $(-1)^k$ if $n = 2k+1$, so the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

**17.** $\sum_{n=1}^{\infty} (-1)^n \sin \frac{\pi}{n}$. $b_n = \sin \frac{\pi}{n} > 0$ for $n \geq 2$ and $\sin \frac{\pi}{n} \geq \sin \frac{\pi}{n+1}$, and $\lim_{n\to\infty} \sin \frac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.

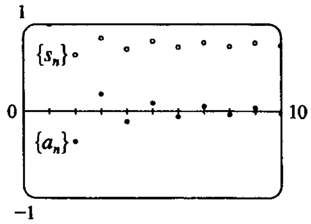
**18.** $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

**19.** $\frac{n^n}{n!} = \frac{n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \geq n \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n^n}{n!} = \infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.

**20.** $\sum_{n=1}^{\infty}\left(-\frac{n}{5}\right)^n$ diverges by the Test for Divergence since $\lim_{n\to\infty}\left(\frac{n}{5}\right)^n=\infty \Rightarrow \lim_{n\to\infty}\left(-\frac{n}{5}\right)^n$ does not exist.

**21.**

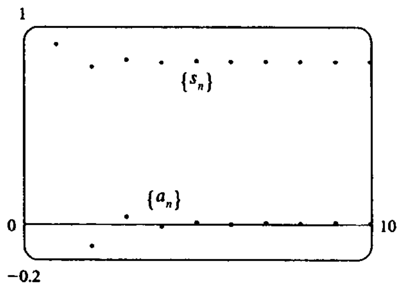
| $n$ | $a_n$ | $s_n$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | −0.35355 | 0.64645 |
| 3 | 0.19245 | 0.83890 |
| 4 | −0.125 | 0.71390 |
| 5 | 0.08944 | 0.80334 |
| 6 | −0.06804 | 0.73530 |
| 7 | 0.05399 | 0.78929 |
| 8 | −0.04419 | 0.74510 |
| 9 | 0.03704 | 0.78214 |
| 10 | −0.03162 | 0.75051 |




By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{3/2}}\approx 0.75051$ is $|s-s_{10}|\le b_{11}=1/(11)^{3/2}\approx 0.0275$ (to four decimal places, rounded up).

**22.**

| $n$ | $a_n$ | $s_n$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | −0.125 | 0.875 |
| 3 | 0.03704 | 0.91204 |
| 4 | −0.01563 | 0.89641 |
| 5 | 0.008 | 0.90441 |
| 6 | −0.00463 | 0.89978 |
| 7 | 0.00292 | 0.90270 |
| 8 | −0.00195 | 0.90074 |
| 9 | 0.00137 | 0.90212 |
| 10 | −0.001 | 0.90112 |




By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\approx 0.90112$ is $|s-s_{10}|\le b_{11}=1/11^3\approx 0.0007513$.

**23.** The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^2}<\frac{1}{n^2}$ and

(ii) $\lim_{n\to\infty}\frac{1}{n^2}=0$, so the series is convergent. Now $b_{10}=\frac{1}{10^2}=0.01$ and $b_{11}=\frac{1}{11^2}=\frac{1}{121}\approx 0.008<0.01$, so by the Alternating Series Estimation Theorem, $n=10$. (That is, since the 11th term is less than the desired error, we need to add the first 10 terms to get the sum to the desired accuracy.)

**24.** The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^4}<\frac{1}{n^4}$ and

(ii) $\lim_{n\to\infty}\frac{1}{n^4}=0$, so the series is convergent. Now $b_5=1/5^4=0.0016>0.001$ and $b_6=1/6^4\approx 0.00077<0.001$, so by the Alternating Series Estimation Theorem, $n=5$.

**25.** The series $\sum_{n=1}^{\infty}$

because $b_{n+1}$

$\lim_{n\to\infty}\frac{2^n}{n!}=\frac{2}{n}$

$b_8=2^8/8!\approx$

is less than the

**26.** The series $\sum_{n=1}^{\infty}$

$b_{n+1}=\frac{n+1}{4^{n+1}}$

$b_5=5/4^5\approx$

Theorem, $n=$

**27.** $b_7=\frac{1}{7^5}=$

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}$

not change the

**28.** $b_6=\frac{6}{8^6}=$

$\sum_{n=1}^{\infty}\frac{(-1)^n n}{8^n}$

change the fou

**29.** $b_7=\frac{7^2}{10^7}=$

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{10^n}$

Adding $b_7$ to $s$

places, is 0.06

**30.** $b_6=\frac{1}{3^6\cdot 6!}=$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n n!}\approx$

change the fou