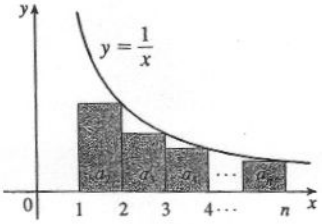
37. (a) From the figure, $a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx$, so with

$f(x) = \frac{1}{x}$, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le \int_1^n \frac{1}{x}\,dx = \ln n$. Thus,

$_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le 1 + \ln n$.




(b) By part (a), $s_{10^6} \le 1 + \ln 10^6 \approx 14.82 < 15$ and $s_{10^9} \le 1 + \ln 10^9 \approx 21.72 < 22$.

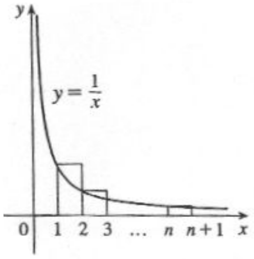
(a) The sum of the areas of the $n$ rectangles in the graph to the right is

$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Now $\int_1^{n+1} \frac{dx}{x}$ is less than this sum because

the rectangles extend above the curve $y = 1/x$, so

$\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and since

$\ln n < \ln(n+1)$, $0 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n = t_n$.



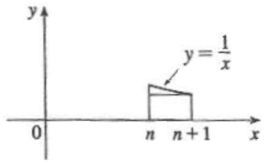

(b) The area under $f(x) = 1/x$ between $x = n$ and $x = n+1$ is

$\int_n^{n+1} \frac{dx}{x} = \ln(n+1) - \ln n$, and this is clearly greater than the

area of the inscribed rectangle in the figure to the right

$\left[\text{which is } \frac{1}{n+1}\right]$, so $t_n - t_{n+1} = [\ln(n+1) - \ln n] - \frac{1}{n+1} > 0$,

and so $t_n > t_{n+1}$, so $\{t_n\}$ is a decreasing sequence.




(c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all $n$. Thus, $0 < t_n \le t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by Theorem 12.1.11.

39. $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \frac{1}{n^{-\ln b}}$. This is a $p$-series, which converges for all $b$ such that $-\ln b > 1$

$\Leftrightarrow \ln b < -1 \Leftrightarrow b < e^{-1} \Leftrightarrow b < 1/e$ [with $b > 0$].

## 12.4 The Comparison Tests

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all $n$ and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 2.)

   (b) If $a_n < b_n$ for all $n$, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

2. (a) If $a_n > b_n$ for all $n$, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

   (b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all $n$ and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. $\frac{1}{n^2+n+1} <$
because it is a

4. $\frac{2}{n^3+4} < \frac{2}{n^3}$
because it is a c

5. $\frac{5}{2+3^n} < \frac{5}{3^n}$
because $\sum_{n=1}^{\infty} \frac{1}{3^n}$

6. $\frac{1}{n-\sqrt{n}} > \frac{1}{n}$
$\sum_{n=2}^{\infty} \frac{1}{n}$.

7. $\frac{n+1}{n^2} > \frac{n}{n^2} =$

8. $\frac{4+3^n}{2^n} > \frac{3^n}{2^n}$
$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.

9. $\frac{\cos^2 n}{n^2+1} \le \frac{}{n^2}$
($p = 2 > 1$).

10. $\frac{n^2-1}{3n^4+1} < \frac{}{3}$
because it is a
$n > 1$, which i

11. If $a_n = \frac{n^2+}{n^3-}$
the Limit Com
*Or:* Since $a_n =$

12. $\frac{1+\sin n}{10^n} \le$
multiple of a c

13. $\frac{n-1}{n4^n}$ is posit
geometric seri

**3.** $\dfrac{1}{n^2+n+1} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+n+1}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a $p$-series with $p = 2 > 1$.

**4.** $\dfrac{2}{n^3+4} < \dfrac{2}{n^3}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3+4}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3} = 2\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, which converges because it is a constant multiple of a convergent $p$-series ($p = 3 > 1$).

**5.** $\dfrac{5}{2+3^n} < \dfrac{5}{3^n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{5}{2+3^n}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{5}{3^n} = 5\sum\limits_{n=1}^{\infty} \dfrac{1}{3^n}$, which converges because $\sum\limits_{n=1}^{\infty} \dfrac{1}{3^n}$ is a convergent geometric series with $r = \frac{1}{3}$ ($|r| < 1$).

**6.** $\dfrac{1}{n-\sqrt{n}} > \dfrac{1}{n}$ for all $n \geq 2$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n-\sqrt{n}}$ diverges by comparison with the divergent (partial) harmonic series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$.

**7.** $\dfrac{n+1}{n^2} > \dfrac{n}{n^2} = \dfrac{1}{n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n^2}$ diverges by comparison with the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$.

**8.** $\dfrac{4+3^n}{2^n} > \dfrac{3^n}{2^n} = \left(\dfrac{3}{2}\right)^n$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.

**9.** $\dfrac{\cos^2 n}{n^2+1} \leq \dfrac{1}{n^2+1} < \dfrac{1}{n^2}$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{\cos^2 n}{n^2+1}$ converges by comparison with the $p$-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ ($p = 2 > 1$).

**10.** $\dfrac{n^2-1}{3n^4+1} < \dfrac{n^2}{3n^4+1} < \dfrac{n^2}{3n^4} = \dfrac{1}{3}\dfrac{1}{n^2}$. $\sum\limits_{n=1}^{\infty} \dfrac{n^2-1}{3n^4+1}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{3n^2}$, which converges because it is a constant multiple of a convergent $p$-series ($p = 2 > 1$). The terms of the given series are positive for $n > 1$, which is good enough.

**11.** If $a_n = \dfrac{n^2+1}{n^3-1}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3+n}{n^3-1} = \lim\limits_{n\to\infty} \dfrac{1+1/n^2}{1-1/n^3} = 1$, so $\sum\limits_{n=2}^{\infty} \dfrac{n^2+1}{n^3-1}$ diverges by the Limit Comparison Test with the divergent (partial) harmonic series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$.

*Or:* Since $a_n = \dfrac{n^2+1}{n^3-1} > \dfrac{n^2+1}{n^3} > \dfrac{n^2}{n^3} = \dfrac{1}{n} = b_n$, we could use the Comparison Test.

**12.** $\dfrac{1+\sin n}{10^n} \leq \dfrac{2}{10^n}$ and $\sum\limits_{n=0}^{\infty} \dfrac{2}{10^n} = 2\sum\limits_{n=0}^{\infty} \left(\dfrac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.

**13.** $\dfrac{n-1}{n4^n}$ is positive for $n > 1$ and $\dfrac{n-1}{n4^n} < \dfrac{n}{n4^n} = \dfrac{1}{4^n}$, so $\sum\limits_{n=1}^{\infty} \dfrac{n-1}{n4^n}$ converges by comparison with the convergent geometric series $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{4}\right)^n$.

**14.** $\dfrac{\sqrt{n}}{n-1} > \dfrac{\sqrt{n}}{n} = \dfrac{1}{\sqrt{n}}$, so $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) $p$-series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ $\left(p = \frac{1}{2} \le 1\right)$.

**15.** $\dfrac{2+(-1)^n}{n\sqrt{n}} \le \dfrac{3}{n\sqrt{n}}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ converges because it is a constant multiple of the convergent $p$-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ $\left(p = \frac{3}{2} > 1\right)$, so the given series converges by the Comparison Test.

**16.** $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent $p$-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $\left(p = \frac{3}{2} > 1\right)$.

**17.** Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

**18.** Use the Limit Comparison Test with $a_n = \dfrac{1}{2n+3}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n+3} = \lim_{n\to\infty} \frac{1}{2+(3/n)} = \frac{1}{2} > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$.

**19.** $\dfrac{2^n}{1+3^n} < \dfrac{2^n}{3^n} = \left(\dfrac{2}{3}\right)^n$. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series $\left(|r| = \frac{2}{3} < 1\right)$, so $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{1+3^n}$ converges by the Comparison Test.

**20.** Use the Limit Comparison Test with $a_n = \dfrac{1+2^n}{1+3^n}$ and $b_n = \dfrac{2^n}{3^n}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(1/2)^n+1}{(1/3)^n+1} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges (geometric series with $|r| = \frac{2}{3} < 1$), $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ also converges.

**21.** Use the Limit Comparison Test with $a_n = \dfrac{1}{1+\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent $p$-series $\left(p = \frac{1}{2} \le 1\right)$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ also diverges.

**22.** Use the Limit Comparison Test with $a_n = \dfrac{n+2}{(n+1)^3}$ and $b_n = \dfrac{1}{n^2}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+2)}{(n+1)^3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1 > 0$. Since $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ is a convergent (partial) $p$-series $(p = 2 > 1)$, the series $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ also converges.

**23.** Use the Limit

$\lim_{n\to\infty} \frac{a_n}{b_n} =$ li

convergent $p$-s

**24.** If $a_n = \frac{n^2 -}{n^3 +}$

$\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+}$

$a_n > 0$ for $n$

**25.** If $a_n = \frac{1+}{\sqrt{1-}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

Limit Compar

**26.** If $a_n = \frac{n -}{\sqrt[3]{n^7}}$

so $\sum_{n=1}^{\infty} \frac{n+}{\sqrt[3]{n^7}-}$

**27.** Use the Limit

Since $\sum_{n=1}^{\infty} e^{-}$

converges.

**28.** Use the Limit

$\lim_{n\to\infty} \frac{a_n}{b_n} =$

$\sum_{n=1}^{\infty} \frac{2n^2}{3^n(n^2+}$

**29.** Clearly $n! =$

geometric se

**30.** $\frac{n!}{n^n} = \frac{1\cdot 2\cdot}{n\cdot}$

converges als

**23.** Use the Limit Comparison Test with $a_n = \dfrac{5+2n}{(1+n^2)^2}$ and $b_n = \dfrac{1}{n^3}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n\to\infty}\frac{5n^3+2n^4}{(1+n^2)^2}\cdot\frac{1/n^4}{1/(n^2)^2} = \lim_{n\to\infty}\frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0.$$ Since $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent $p$-series ($p = 3 > 1$), the series $\sum_{n=1}^{\infty}\frac{5+2n}{(1+n^2)^2}$ also converges.

**24.** If $a_n = \dfrac{n^2-5n}{n^3+n+1}$ and $b_n = \dfrac{1}{n}$, then $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\dfrac{n^3-5n^2}{n^3+n+1} = \lim_{n\to\infty}\dfrac{1-5/n}{1+1/n^2+1/n^3} = 1 > 0$, so $\sum_{n=1}^{\infty}\dfrac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$. (Note that $a_n > 0$ for $n \geq 6$.)

**25.** If $a_n = \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \dfrac{1}{n}$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n\to\infty}\frac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1 > 0,$$ so $\sum_{n=1}^{\infty}\dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$.

**26.** If $a_n = \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}}\cdot\frac{n^{-7/3}}{n^{-7/3}} = \lim_{n\to\infty}\frac{1+5/n}{[(n^7+n^2)/n^7]^{1/3}}$$

$$= \lim_{n\to\infty}\frac{1+5/n}{(1+1/n^5)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1 > 0,$$

so $\sum_{n=1}^{\infty}\dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent $p$-series $\sum_{n=1}^{\infty}\dfrac{1}{n^{4/3}}$.

**27.** Use the Limit Comparison Test with $a_n = \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$: $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^2 = 1 > 0$.

Since $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty}\dfrac{1}{e^n}$ is a convergent geometric series $\left(|r| = \frac{1}{e} < 1\right)$, the series $\sum_{n=1}^{\infty}\left(1+\dfrac{1}{n}\right)^2 e^{-n}$ also converges.

**28.** Use the Limit Comparison Test with $a_n = \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ and $b_n = \dfrac{1}{3^n}$.

$\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\dfrac{2n^2+7n}{n^2+5n-1} = 2 > 0$, and since $\sum_{n=1}^{\infty} b_n$ is a convergent geometric series ($|r| = \frac{1}{3} < 1$), $\sum_{n=1}^{\infty}\dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ converges also.

**29.** Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2\cdot 2\cdot 2\cdot\cdots\cdot 2\cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$. $\sum_{n=1}^{\infty}\dfrac{1}{2^{n-1}}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), so $\sum_{n=1}^{\infty}\dfrac{1}{n!}$ converges by the Comparison Test.

**30.** $\dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)n}{n\cdot n\cdot n\cdot\cdots\cdot n\cdot n} \leq \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot 1\cdot 1\cdot\cdots\cdot 1$ for $n \geq 2$, so since $\sum_{n=1}^{\infty}\dfrac{2}{n^2}$ converges ($p = 2 > 1$), $\sum_{n=1}^{\infty}\dfrac{n!}{n^n}$ converges also by the Comparison Test.

**31.** Use the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} \sin(1/n)$ also diverges. (Note that we could also use l'Hospital's Rule to evaluate the limit:
$\lim\limits_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{H}{=} \lim\limits_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim\limits_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1$.)

**32.** Use the Limit Comparison Test with $a_n = \frac{1}{n^{1+1/n}}$ and $b_n = \frac{1}{n}$. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n}{n^{1+1/n}} = \lim\limits_{n\to\infty} \frac{1}{n^{1/n}} = 1$ (since $\lim\limits_{x\to\infty} x^{1/x} = 1$ by l'Hospital's Rule), so $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) $\Rightarrow$ $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.

**33.** $\sum\limits_{i=1}^{10} \frac{1}{n^4+n^2} = \frac{1}{2} + \frac{1}{20} + \frac{1}{90} + \cdots + \frac{1}{10{,}100} \approx 0.567975$. Now $\frac{1}{n^4+n^2} < \frac{1}{n^4}$, so using the reasoning and notation of Example 5, the error is $R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \frac{1}{n^4} \le \int_{10}^{\infty} \frac{dx}{x^4} = \lim\limits_{t\to\infty} \left[-\frac{x^{-3}}{3}\right]_{10}^{t} = \frac{1}{3000} = 0.000\overline{3}$.

**34.** $\sum\limits_{n=1}^{10} \frac{1+\cos n}{n^5} = 1 + \cos 1 + \frac{1+\cos 2}{32} + \frac{1+\cos 3}{243} + \cdots + \frac{1+\cos 10}{100{,}000} \approx 1.55972$. Now $\frac{1+\cos n}{n^5} \le \frac{2}{n^5}$, so as in Example 5, $R_{10} \le T_{10} \le \int_{10}^{\infty} \frac{2}{x^5}\,dx = 2 \lim\limits_{t\to\infty} \left[-\frac{1}{4}x^{-4}\right]_{10}^{t} = 0.00005$.

**35.** $\sum\limits_{n=1}^{10} \frac{1}{1+2^n} = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \cdots + \frac{1}{1025} \approx 0.76352$. Now $\frac{1}{1+2^n} < \frac{1}{2^n}$, so the error is
$R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \frac{1}{2^n} = \frac{1/2^{11}}{1-1/2}$ (geometric series) $\approx 0.00098$.

**36.** $\sum\limits_{n=1}^{10} \frac{n}{(n+1)3^n} = \frac{1}{6} + \frac{2}{27} + \frac{3}{108} + \cdots + \frac{10}{649{,}539} \approx 0.283597$. Now $\frac{n}{(n+1)3^n} < \frac{n}{n\cdot 3^n} = \frac{1}{3^n}$, so the error is
$R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \frac{1}{3^n} = \frac{1/3^{11}}{1-1/3} \approx 0.0000085$.

**37.** Since $\frac{d_n}{10^n} \le \frac{9}{10^n}$ for each $n$, and since $\sum\limits_{n=1}^{\infty} \frac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$), $0.d_1d_2d_3\ldots = \sum\limits_{n=1}^{\infty} \frac{d_n}{10^n}$ will always converge by the Comparison Test.

**38.** Clearly, if $p < 0$ then the series diverges, since $\lim\limits_{n\to\infty} \frac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n$ $\Rightarrow$ $\frac{1}{n^p \ln n} \ge \frac{1}{n \ln n}$ and $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (Exercise 12.3.21), so $\sum\limits_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison Test with $a_n = \frac{1}{n^p \ln n}$ and $b_n = \frac{1}{n^p}$. $\sum\limits_{n=2}^{\infty} b_n$ converges, and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{1}{\ln n} = 0$, so $\sum\limits_{n=2}^{\infty} \frac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

**39.** Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$, so there exists $N$ such that $|a_n - 0| < 1$ for all $n > N$ $\Rightarrow$ $0 \le a_n < 1$ for all $n > N$ $\Rightarrow$ $0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

**40.** (a) Since $\lim\limits_{n\to\infty}$
since $a_n$ a
(b) (i) If $a_n$
conve
(ii) If $a_n$
Now
**41.** (a) Since $\lim\limits_{n\to\infty}$
Definition
Comparis
(b) (i) If $a_n$
$\lim\limits_{n\to\infty}$
(ii) If $a_n$
$\lim\limits_{n\to\infty}$
**42.** Let $a_n = \frac{1}{n^2}$
**43.** $\lim\limits_{n\to\infty} na_n =$
either both se
Therefore, $\sum$
**44.** First we obse
$\lim\limits_{n\to\infty} a_n = 0$
$\sum a_n$ is conv
**45.** Yes. Since $\sum$
$\sum b_n = \sum$ s
$\lim\limits_{n\to\infty} \frac{b_n}{a_n} =$
Test.
**46.** Yes. Since $\sum$
then $\sum_{n=1}^{\infty} a$
the second te