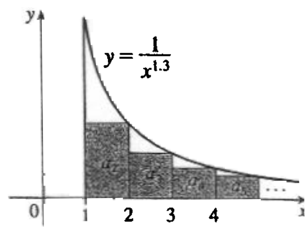
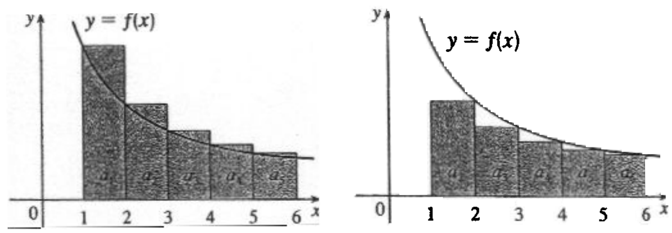
12.3 The Integral Test and Estimates of Sums



1. The picture shows that $a_2 = \frac{1}{2^{1.3}} < \int_1^2 \frac{1}{x^{1.3}}\,dx$,

$a_3 = \frac{1}{3^{1.3}} < \int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (8.8.2) with $p = 1.3 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^5 a_i$. From the second figure, we see that $\sum_{i=2}^6 a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^6 a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^5 a_i$.

3. The function $f(x) = 1/x^4$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_1^t x^{-4}\,dx = \lim_{t\to\infty}\left[\frac{x^{-3}}{-3}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{3t^3}+\frac{1}{3}\right) = \frac{1}{3}$. Since this improper integral is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is also convergent by the Integral Test.

4. The function $f(x) = 1/\sqrt[4]{x} = x^{-1/4}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\int_1^{\infty} x^{-1/4}\,dx = \lim_{t\to\infty} \int_1^t x^{-1/4}\,dx = \lim_{t\to\infty}\left[\frac{4}{3}x^{3/4}\right]_1^t = \lim_{t\to\infty}\left(\frac{4}{3}t^{3/4} - \frac{4}{3}\right) = \infty$, so $\sum_{n=1}^{\infty} 1/\sqrt[4]{n}$ diverges.

5. The function $f(x) = 1/(3x+1)$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{dx}{3x+1} = \lim_{b\to\infty}\int_1^b \frac{dx}{3x+1} = \lim_{b\to\infty}\left[\tfrac{1}{3}\ln(3x+1)\right]_1^b = \lim_{b\to\infty}\left[\tfrac{1}{3}\ln(3b+1) - \tfrac{1}{3}\ln 4\right] = \infty$$

so the improper integral diverges, and so does the series $\sum_{n=1}^{\infty} 1/(3n+1)$.

6. The function $f(x) = e^{-x}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_1^b = \lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right) = e^{-1}$, so $\sum_{n=1}^{\infty} e^{-n}$ converges. *Note:* This is a geometric series, with first term $a = e^{-1}$ and ratio $r = e^{-1}$. Since $|r| < 1$, the series converges to $e^{-1}/(1-e^{-1}) = 1/(e-1)$.

7. $f(x) = xe^{-x}$ is continuous and positive on $[1,\infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) < 0$ for $x > 1$, so f is decreasing on $[1,\infty)$. Thus, the Integral Test applies.

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b \quad \text{(by parts)}$$

$$= \lim_{b\to\infty}\left[-be^{-b} - e^{-b} + e^{-1} + e^{-1}\right] = 2/e$$

since $\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty}(b/e^b) \overset{H}{=} \lim_{b\to\infty}(1/e^b) = 0$ and $\lim_{b\to\infty} e^{-b} = 0$. Thus, $\sum_{n=1}^{\infty} ne^{-n}$ converges.

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8. The function $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\displaystyle\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \left(1 + \frac{1}{x+1}\right) dx = \lim_{t\to\infty} \left[x + \ln(x+1)\right]_1^t = \lim_{t\to\infty} (t + \ln(t+1) - 1 - \ln 2) = \infty$, so $\displaystyle\int_1^\infty \frac{x+2}{x+1}\,dx$ is divergent and the series $\displaystyle\sum_{n=1}^\infty \frac{n+2}{n+1}$ is divergent. NOTE: $\displaystyle\lim_{n\to\infty} \frac{n+2}{n+1} = 1$, so the given series diverges by the Test for Divergence.

9. The series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{0.85}}$ is a p-series with $p = 0.85 \le 1$, so it diverges by (1). Therefore, the series $\displaystyle\sum_{n=1}^\infty \frac{2}{n^{0.85}}$ must also diverge, for if it converged, then $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{0.85}}$ would have to converge (by Theorem 8(i) in Section 11.2).

10. $\displaystyle\sum_{n=1}^\infty n^{-1.4}$ and $\displaystyle\sum_{n=1}^\infty n^{-1.2}$ are p-series with $p > 1$, so they converge by (1). Thus, $\displaystyle\sum_{n=1}^\infty 3n^{-1.2}$ converges by Theorem 8(i) in Section 11.2. It follows from Theorem 8(ii) that the given series $\displaystyle\sum_{n=1}^\infty (n^{-1.4} + 3n^{-1.2})$ also converges.

11. $1 + \dfrac{1}{8} + \dfrac{1}{27} + \dfrac{1}{64} + \dfrac{1}{125} + \cdots = \displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

12. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \dfrac{1}{5\sqrt{5}} + \cdots = \displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n}} = \sum_{n=1}^\infty \frac{1}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$, so it converges by (1).

13. $\displaystyle\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3} = 5\sum_{n=1}^\infty \frac{1}{n^3} - 2\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ by Theorem 12.2.8, since $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$ and $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ both converge by (1) (with $p = 3 > 1$ and $p = \frac{5}{2} > 1$). Thus, $\displaystyle\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3}$ converges.

14. The function $f(x) = \dfrac{5}{x-2}$ is continuous, positive, and decreasing on $[3, \infty)$, so we can apply the Integral Test.

$\displaystyle\int_3^\infty \frac{5}{x-2}\,dx = \lim_{t\to\infty} \int_3^t \frac{5}{x-2}\,dx = \lim_{t\to\infty} \left[5\ln(x-2)\right]_3^t = \lim_{t\to\infty} \left[5\ln(t-2) - 0\right] = \infty$, so the series $\displaystyle\sum_{n=3}^\infty \frac{5}{n-2}$ diverges.

15. The function $f(x) = \dfrac{1}{x^2+4}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can apply the Integral Test.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2+4}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty} \left[\tan^{-1}\left(\frac{t}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right] \\ &= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]\end{aligned}$$

Therefore, the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+4}$ converges.

16. The function $f(x) = \dfrac{3x+2}{x(x+1)} = \dfrac{2}{x} + \dfrac{1}{x+1}$ [by partial fractions] is continuous, positive, and decreasing on $[1, \infty)$ since it is the sum of two such functions. Thus, we can apply the Integral Test.

$$\int_1^\infty \frac{3x+2}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t \left[\frac{2}{x} + \frac{1}{x+1}\right] dx = \lim_{t\to\infty}\left[2\ln x + \ln(x+1)\right]_1^t$$
$$= \lim_{t\to\infty}\left[2\ln t + \ln(t+1) - \ln 2\right] = \infty$$

Thus, the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$ diverges.

17. $f(x) = \dfrac{x}{x^2+1}$ is continuous and positive on $[1, \infty)$, and since $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$, f is also decreasing. Using the Integral Test, $\displaystyle\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\left[\frac{\ln(x^2+1)}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+1) - \ln 2\right] = \infty$, so the series diverges.

18. The function $f(x) = \dfrac{1}{x^2-4x+5} = \dfrac{1}{(x-2)^2+1}$ is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies. $\displaystyle\int_2^\infty f(x)\,dx = \lim_{t\to\infty}\int_2^t f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(x-2)^2+1}\,dx = \lim_{t\to\infty}\left[\tan^{-1}(x-2)\right]_2^t =$ $\displaystyle\lim_{t\to\infty}\left[\tan^{-1}(t-2) - \tan^{-1} 0\right] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$, so the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-4n+5}$ converges. Of course this means that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2-4n+5}$ converges too.

19. $f(x) = xe^{-x^2}$ is continuous and positive on $[1, \infty)$, and since $f'(x) = e^{-x^2}(1-2x^2) < 0$ for $x > 1$, f is decreasing as well. Thus, we can use the Integral Test. $\int_1^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^t = 0 - \left(-\frac{1}{2}e^{-1}\right) = 1/(2e)$. Since the integral converges, the series

20. $f(x) = \dfrac{\ln x}{x^2}$ is continuous and positive for $x \ge 2$, and $f'(x) = \dfrac{1-2\ln x}{x^3} < 0$ for $x \ge 2$, so f is $\displaystyle\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^t$ [by parts] $\overset{H}{=} 1$. Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2} = \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ converges by the Integral Test.

21. $f(x) = \dfrac{1}{x\ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the series diverges.

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