12.2 Series

ET 11.2

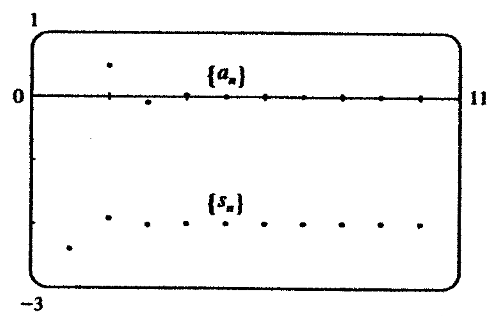
1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum_{i=1}^{n} a_i$.

3.

n	s_n
1	-2.40000
2	-1.92000
3	-2.01600
4	-1.99680
5	-2.00064
6	-1.99987
7	-2.00003
8	-1.99999
9	-2.00000
10	-2.00000



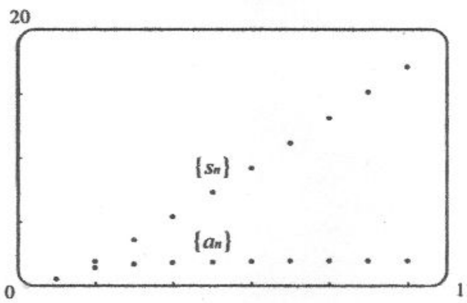
From the graph and the table, it seems that the series converges to -2. In fact, it is a geometric series with $a = -2.4$ and $r = -\frac{1}{5}$, so its sum is

$$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{-2.4}{1-\left(-\frac{1}{5}\right)} = \frac{-2.4}{1.2} = -2.$$ Note that the dot corresponding to $n = 1$ is part of both $\{a_n\}$ and $\{s_n\}$.

TI-86 Note: To graph $\{a_n\}$ and $\{s_n\}$, set your calculator to Param mode and DrawDot mode. (DrawDot is under GRAPH, MORE, FORMT (F3).) Now under `E(t)=` make the assignments: `xt1=t, yt1=12/(-5)^t, xt2=t, yt2=sum seq(yt1,t,1,t,1)`. (sum and seq are under LIST, OPS (F5), MORE.) Under WIND use `1,10,1,0,10,1,-3,1,1` to obtain a graph similar to the one above. Then use TRACE (F4) to see the values.

4.

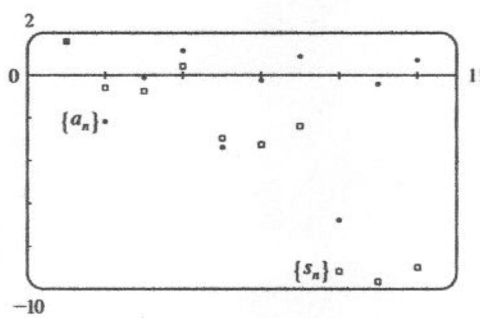
n	s_n
1	0.50000
2	1.90000
3	3.60000
4	5.42353
5	7.30814
6	9.22706
7	11.16706
8	13.12091
9	15.08432
10	17.05462



The series $\sum_{n=1}^{\infty} \frac{2n^2-1}{n^2+1}$ diverges, since its terms do not approach 0.

5.

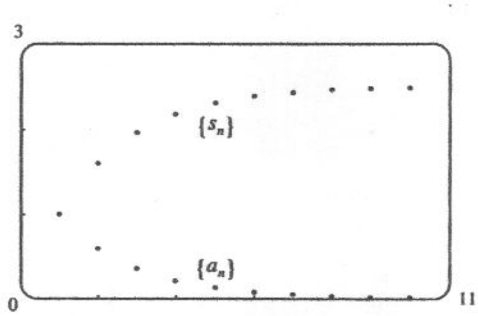
n	s_n
1	1.55741
2	−0.62763
3	−0.77018
4	0.38764
5	−2.99287
6	−3.28388
7	−2.41243
8	−9.21214
9	−9.66446
10	−9.01610



The series $\sum_{n=1}^{\infty} \tan n$ diverges, since its terms do not approach 0.

6.

n	s_n
1	1.00000
2	1.60000
3	1.96000
4	2.17600
5	2.30560
6	2.38336
7	2.43002
8	2.45801
9	2.47481
10	2.48488

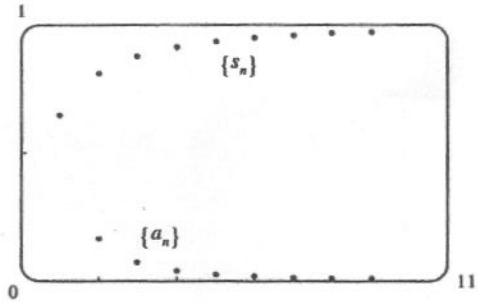


From the graph and the table, it seems that the series converges to 2.5. In fact, it is a geometric series with $a = 1$ and $r = 0.6$, so its sum is

$$\sum_{n=1}^{\infty} (0.6)^{n-1} = \frac{1}{1-0.6} = \frac{1}{2/5} = 2.5.$$

7.

n	s_n
1	0.64645
2	0.80755
3	0.87500
4	0.91056
5	0.93196
6	0.94601
7	0.95581
8	0.96296
9	0.96838
10	0.97259



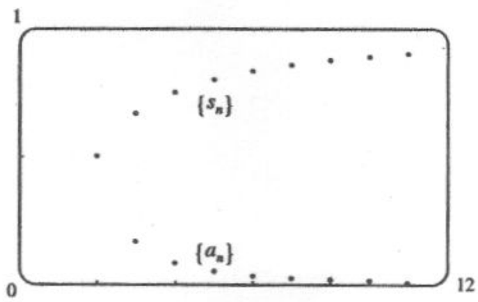
From the graph, it seems that the series converges to 1. To find the sum, we write

$$s_n = \sum_{i=1}^{n} \left(\frac{1}{i^{1.5}} - \frac{1}{(i+1)^{1.5}} \right)$$
$$= \left(1 - \frac{1}{2^{1.5}}\right) + \left(\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}}\right) + \left(\frac{1}{3^{1.5}} - \frac{1}{4^{1.5}}\right) + \cdots + \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}\right) = 1 - \frac{1}{(n+1)^{1.5}}$$

So the sum is $\lim_{n\to\infty} s_n = 1 - 0 = 1$.

8.

n	s_n
2	0.50000
3	0.66667
4	0.75000
5	0.80000
6	0.83333
7	0.85714
8	0.87500
9	0.88889
10	0.90000
11	0.90909
100	0.99000



From the graph and the table, it seems that the series converges to 1. To find the sum, we write

$$s_n = \sum_{i=2}^{n} \frac{1}{i(i-1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i} \right) \qquad \text{[partial fractions]}$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n},$$

and so the sum is $\lim_{n\to\infty} s_n = 1 - 0 = 1$.

9. (a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (12.1.1) [ET (11.1.1)].

(b) Since $\lim\limits_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum\limits_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence (7).

10. (a) Both $\sum\limits_{i=1}^{n} a_i$ and $\sum\limits_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the nth partial sum.

(b) $\sum\limits_{i=1}^{n} a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum\limits_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

11. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$ is a geometric series with first term $a = 3$ and common ratio $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-2/3} = \frac{3}{1/3} = 9$.

12. $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$ is a geometric series with $r = -2$. Since $|r| = 2 > 1$, the series diverges.

13. $-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$ is a geometric series with $a = -2$ and $r = \frac{5/2}{-2} = -\frac{5}{4}$. Since $|r| = \frac{5}{4} > 1$, the series diverges by (4).

14. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.

15. $\sum\limits_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a = 5$ and $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$.

16. $\sum\limits_{n=1}^{\infty} \dfrac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.

17. $\sum\limits_{n=1}^{\infty} \dfrac{(-3)^{n-1}}{4^n} = \dfrac{1}{4} \sum\limits_{n=1}^{\infty} \left(-\dfrac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

18. $\sum\limits_{n=0}^{\infty} \dfrac{1}{(\sqrt{2})^n}$ is a geometric series with ratio $r = \dfrac{1}{\sqrt{2}}$. Since $|r| = \dfrac{1}{\sqrt{2}} < 1$, the series converges. Its sum is

$$\frac{1}{1 - 1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}(\sqrt{2}+1) = 2 + \sqrt{2}.$$

19. $\sum\limits_{n=0}^{\infty} \dfrac{\pi^n}{3^{n+1}} = \dfrac{1}{3} \sum\limits_{n=0}^{\infty} \left(\dfrac{\pi}{3}\right)^n$ is a geometric series with ratio $r = \dfrac{\pi}{3}$. Since $|r| > 1$, the series diverges.

20. $\sum\limits_{n=1}^{\infty} \dfrac{e^n}{3^{n-1}} = 3 \sum\limits_{n=1}^{\infty} \left(\dfrac{e}{3}\right)^n$ is a geometric series with first term $3(e/3) = e$ and ratio $r = \dfrac{e}{3}$. Since $|r| < 1$, the series converges. Its sum is $\dfrac{e}{1 - e/3} = \dfrac{3e}{3 - e}$.

21. $\sum\limits_{n=1}^{\infty} \dfrac{n}{n+5}$ diverges since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n}{n+5} = 1 \neq 0$. [Use (7), the Test for Divergence.]

22. $\sum_{n=1}^{\infty} \frac{3}{n} = 3 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. [If $\sum_{n=1}^{\infty} \frac{3}{n}$ were to converge, then $\sum_{n=1}^{\infty} \frac{1}{n}$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

23. Using partial fractions, the partial sums are

$$s_n = \sum_{i=2}^{n} \frac{2}{(i-1)(i+1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1}\right)$$
$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right)$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

Thus, $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right) = \frac{3}{2}$.

24. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim_{n\to\infty} \left(1 + \frac{1}{n^2+2n}\right) = 1 \neq 0.$

25. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1 \neq 0.$

26. Converges. $s_n = \sum_{i=1}^{n} \frac{2}{i^2+4i+3} = \sum_{i=1}^{n} \left(\frac{1}{i+1} - \frac{1}{i+3}\right)$ (using partial fractions). The latter sum is

$\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$

(telescoping series). Thus, $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

27. Converges. $\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{3^n}{6^n} + \frac{2^n}{6^n}\right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right] = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

28. $\sum_{n=1}^{\infty} \left[(0.8)^{n-1} - (0.3)^n\right] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n$ [difference of two convergent geometric series]

$= \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 5 - \frac{3}{7} = \frac{32}{7}$.

29. $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$ diverges by the Test for Divergence since

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0.$

30. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\dfrac{n}{2n+5}\right) = \lim_{n\to\infty} \ln\left(\dfrac{1}{2+5/n}\right) = \ln\frac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$. Its sum is $\dfrac{\cos 1}{1-\cos 1} \approx 1.175343$.

33. The first series is a telescoping sum:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3}{n(n+3)} &= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3}\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}
\end{aligned}$$

The second series is geometric with first term $\frac{5}{4}$ and ratio $\frac{1}{4}$: $\sum_{n=1}^{\infty} \dfrac{5}{4^n} = \dfrac{5/4}{1-1/4} = \dfrac{5}{3}$. Thus,

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)} + \frac{5}{4^n}\right) = \sum_{n=1}^{\infty}\frac{3}{n(n+3)} + \sum_{n=1}^{\infty}\frac{5}{4^n} \text{ [sum of two convergent series]} = \frac{11}{6} + \frac{5}{3} = \frac{7}{2}.$$

34. $\sum_{n=1}^{\infty}\left(\dfrac{3}{5^n} + \dfrac{2}{n}\right)$ diverges because $\sum_{n=1}^{\infty}\dfrac{2}{n} = 2\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges. (If it converged, then $\frac{1}{2}\cdot 2\sum_{n=1}^{\infty}\frac{1}{n}$ would also converge by Theorem 8(i), but we know from Example 7 that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.) If the given series converges, then the difference $\sum_{n=1}^{\infty}\left(\dfrac{3}{5^n} + \dfrac{2}{n}\right) - \sum_{n=1}^{\infty}\dfrac{3}{5^n}$ must converge (since $\sum_{n=1}^{\infty}\frac{3}{5^n}$ is a convergent geometric series) and equal $\sum_{n=1}^{\infty}\frac{2}{n}$, but we have just seen that $\sum_{n=1}^{\infty}\frac{2}{n}$ diverges, so the given series must also diverge.

35. $0.\overline{2} = \dfrac{2}{10} + \dfrac{2}{10^2} + \cdots$ is a geometric series with $a = \dfrac{2}{10}$ and $r = \dfrac{1}{10}$. It converges to $\dfrac{a}{1-r} = \dfrac{2/10}{1-1/10} = \dfrac{2}{9}$.

36. $0.\overline{73} = \dfrac{73}{10^2} + \dfrac{73}{10^4} + \cdots = \dfrac{73/10^2}{1-1/10^2} = \dfrac{73/100}{99/100} = \dfrac{73}{99}$

37. $3.\overline{417} = 3 + \dfrac{417}{10^3} + \dfrac{417}{10^6} + \cdots = 3 + \dfrac{417/10^3}{1-1/10^3} = 3 + \dfrac{417}{999} = \dfrac{3414}{999} = \dfrac{1138}{333}$

38. $6.2\overline{54} = 6.2 + \dfrac{54}{10^3} + \dfrac{54}{10^5} + \cdots = 6.2 + \dfrac{54/10^3}{1-1/10^2} = \dfrac{62}{10} + \dfrac{54}{990} = \dfrac{6192}{990} = \dfrac{344}{55}$

39. $0.123\overline{456} = \dfrac{123}{1000} + \dfrac{0.000456}{1 - 0.001} = \dfrac{123}{1000} + \dfrac{456}{999{,}000} = \dfrac{123{,}333}{999{,}000} = \dfrac{41{,}111}{333{,}000}$

40. $5.\overline{6021} = 5 + \dfrac{6021}{10^4} + \dfrac{6021}{10^8} + \cdots = 5 + \dfrac{6021/10^4}{1 - 1/10^4} = 5 + \dfrac{6021}{9999} = \dfrac{56{,}016}{9999} = \dfrac{6224}{1111}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$ is a geometric series with $r = \dfrac{x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x|}{3} < 1$ $\Leftrightarrow$ $|x| < 3$; that is, $-3 < x < 3$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{x/3}{1 - x/3} = \dfrac{x/3}{1 - x/3} \cdot \dfrac{3}{3} = \dfrac{x}{3 - x}$.

42. $\sum_{n=1}^{\infty} (x-4)^n$ is a geometric series with $r = x - 4$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x - 4| < 1$ $\Leftrightarrow$ $3 < x < 5$. In that case, the sum of the series is $\dfrac{x-4}{1-(x-4)} = \dfrac{x-4}{5-x}$.

43. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n$ is a geometric series with $r = 4x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $4|x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1-4x}$.

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$ is a geometric series with $r = \dfrac{x+3}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x+3|}{2} < 1$ $\Leftrightarrow$ $|x + 3| < 2$ $\Leftrightarrow$ $-5 < x < -1$. For these values of x, the sum of the series is $\dfrac{1}{1-(x+3)/2} = \dfrac{2}{2-(x+3)} = -\dfrac{2}{x+1}$.

45. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$ is a geometric series with first term 1 and ratio $r = \dfrac{\cos x}{2}$, so it converges $\Leftrightarrow$ $|r| < 1$. But $|r| = \dfrac{|\cos x|}{2} \le \dfrac{1}{2}$ for all x. Thus, the series converges for all real values of x and the sum of the series is $\dfrac{1}{1 - (\cos x)/2} = \dfrac{2}{2 - \cos x}$.

46. Because $\dfrac{1}{n} \to 0$ and ln is continuous, we have $\displaystyle\lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = \ln 1 = 0$. We now show that the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} [\ln(n+1) - \ln n]$ diverges.
$s_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n) = \ln(n+1) - \ln 1 = \ln(n+1)$. As $n \to \infty$, $s_n = \ln(n+1) \to \infty$, so the series diverges.

47. After defining f, We use `convert(f,parfrac);` in Maple, `Apart` in Mathematica, or `Expand Rational` and `Simplify` in Derive to find that the general term is $\dfrac{1}{(4n+1)(4n-3)} = -\dfrac{1/4}{4n+1} + \dfrac{1/4}{4n-3}$. So the