12.10 Taylor and Maclaurin Series

1. Using Theorem 5 with $\sum_{n=0}^{\infty} b_n (x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. (a) Using Formula 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $1.6 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

3.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
⋮	⋮	⋮

We use Equation 7 with $f(x) = \cos x$.

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

If $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = x^2 \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x.$$

So $R = \infty$ (Ratio Test).

4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2 \sin 2x$	0
3	$-2^3 \cos 2x$	-2^3
4	$2^4 \sin 2x$	0
⋮	⋮	⋮

$f^{(n)}(0) = 0$ if n is even and $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$, so

$$\sin 2x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^2\,|x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x,$$

so $R = \infty$ (Ratio Test).

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-3}$	1
1	$-3(1+x)^{-4}$	-3
2	$12(1+x)^{-5}$	12
3	$-60(1+x)^{-6}$	-60
4	$360(1+x)^{-7}$	360
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}(1+x)^{-3} &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 1 - 3x + \frac{4\cdot 3}{2!}x^2 - \frac{5\cdot 4\cdot 3}{3!}x^3 + \frac{6\cdot 5\cdot 4\cdot 3}{4!}x^4 - \cdots \\ &= 1 - 3x + \frac{4\cdot 3\cdot 2}{2\cdot 2!}x^2 - \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3!}x^3 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 4!}x^4 - \cdots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!\,x^n}{2(n!)} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+2)(n+1)x^n}{2}\end{aligned}$$

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+3)(n+2)x^{n+1}}{2}\cdot\dfrac{2}{(n+2)(n+1)x^n}\right| = |x|\lim\limits_{n\to\infty}\dfrac{n+3}{n+1} = |x| < 1$ for convergence, so $R = 1$ (Ratio Test).

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\ln(1+x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &\quad + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\ &= x - \tfrac{1}{2}x^2 + \tfrac{2}{6}x^3 - \tfrac{6}{24}x^4 + \tfrac{24}{120}x^5 - \cdots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n\end{aligned}$$

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{n+1}\cdot\dfrac{n}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{1+1/n} = |x| < 1$ for convergence, so $R = 1$.

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	5^2e^{5x}	25
3	5^3e^{5x}	125
4	5^4e^{5x}	625
$\vdots$	$\vdots$	$\vdots$

$$e^{5x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{5^n}{n!}x^n.$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n|x|^n}\right] \\ &= \lim_{n\to\infty}\frac{5|x|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.\end{aligned}$$

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
$\vdots$	$\vdots$	$\vdots$

$$xe^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{n}{n!}x^n = \sum_{n=1}^{\infty} \frac{n}{n!}x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}.$$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{|x|^{n+1}}{n!} \cdot \frac{(n-1)!}{|x|^n}\right] = \lim_{n\to\infty} \frac{|x|}{n} = 0 < 1$ for all x, so $R = \infty$.

9.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \quad \text{so } \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Use the Ratio Test to find R. If $a_n = \dfrac{x^{2n+1}}{(2n+1)!}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}}\right| = x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0 < 1$$

for all x, so $R = \infty$.

10.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad \text{so } \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Use the Ratio Test to find R. If $a_n = \dfrac{x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}}\right| = x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1$$

for all x, so $R = \infty$.

11.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1+x+x^2$	7
1	$1+2x$	5
2	2	2
3	0	0
4	0	0
$\vdots$	$\vdots$	$\vdots$

$$f(x) = 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 + \sum_{n=3}^{\infty} \frac{0}{n!}(x-2)^n = 7 + 5(x-2) + (x-2)^2$$

Since $a_n = 0$ for large n, $R = \infty$.

12.

n
0
1
2
3
4
5
$\vdots$

13. Clearly, $f^{(n)}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right.$

14.

$f^{(n)}(2) =$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

15.

$\cos x = \sum_{k=0}^{\infty}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

12.

n	$f^{(n)}(x)$	$f^{(n)}(-1)$
0	x^3	-1
1	$3x^2$	3
2	$6x$	-6
3	6	6
4	0	0
5	0	0
$\vdots$	$\vdots$	$\vdots$

$$f(x) = -1 + 3(x+1) - \tfrac{6}{2!}(x+1)^2 + \tfrac{6}{3!}(x+1)^3$$
$$= -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$$

Since $a_n = 0$ for large n, $R = \infty$.

13. Clearly, $f^{(n)}(x) = e^x$, so $f^{(n)}(3) = e^3$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{e^3}{n!}(x-3)^n$. If $a_n = \dfrac{e^3}{n!}(x-3)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

14.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3\cdot 2x^{-4}$	$-\frac{3\cdot 2}{16}$
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(2) = \dfrac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \geq 1$, so $\ln x = \ln 2 + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}(x-2)^n}{n\cdot 2^n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-2|}{2} < 1 \text{ for convergence, so } |x-2| < 2 \ \Rightarrow\ R = 2.$$

15.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
$\vdots$	$\vdots$	$\vdots$

$$\cos x = \sum_{k=0}^{\infty}\frac{f^{(k)}(\pi)}{k!}(x-\pi)^k = -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \cdots = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x-\pi|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi|^{2n}}\right] = \lim_{n\to\infty}\frac{|x-\pi|^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

16.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$\sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!}\left(x - \frac{\pi}{2}\right)^k$$
$$= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x-\pi/2|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi/2|^{2n}}\right] = \lim_{n\to\infty}\frac{|x-\pi/2|^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x,$$

so $R = \infty$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{-1/2}$	$\frac{1}{3}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2}\cdot\frac{1}{3^3}$
2	$\frac{3}{4}x^{-5/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{3^5}$
3	$-\frac{15}{8}x^{-7/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{3^7}$
$\vdots$	$\vdots$	$\vdots$

$$\frac{1}{\sqrt{x}} = \frac{1}{3} - \frac{1}{2\cdot 3^3}(x-9) + \frac{3}{2^2\cdot 3^5}\frac{(x-9)^2}{2!} - \frac{3\cdot 5}{2^3\cdot 3^7}\frac{(x-9)^3}{3!} + \cdots$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)[2(n+1)-1]\,|x-9|^{n+1}}{2^{n+1}\cdot 3^{[2(n+1)+1]}\cdot(n+1)!}\cdot\frac{2^n\cdot 3^{2n+1}\cdot n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,|x-9|^n}\right]$$
$$= \lim_{n\to\infty}\left[\frac{(2n+1)\,|x-9|}{2\cdot 3^2(n+1)}\right] = \frac{1}{9}|x-9| < 1$$

for convergence, so $|x-9| < 9$ and $R = 9$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	-24
4	$120x^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$x^{-2} = 1 - 2(x-1) + 6\cdot\frac{(x-1)^2}{2!} - 24\cdot\frac{(x-1)^3}{3!} + 120\cdot\frac{(x-1)^4}{4!} - \cdots$$
$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - \cdots$$
$$= \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)\,|x-1|^{n+1}}{(n+1)\,|x-1|^n} = \lim_{n\to\infty}\left[\frac{n+2}{n+1}\cdot|x-1|\right] = |x-1| < 1 \text{ for convergence, so } R = 1.$$

19. If $f(x) =$

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20. If $f(x) =$

and $M = 1$

$\lim_{n\to\infty} R_n(x$

21. If $f(x) =$ s

have $|f^{(n+1}$

so by Formu

as $n \to \infty$ f

represents si

22. If $f(x) =$ cc

have $|f^{(n+1)}$

so by Formu

as $n \to \infty$ fc

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23. $\cos x = \sum_{n=0}^{\infty}$ (

24. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

25. $\tan^{-1} x = \sum$

26. $\sin x = \sum_{n=0}^{\infty}$ (-

27. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

28. $\cos x = \sum_{n=0}^{\infty}$ (

$f(x) = x\cos$

29. $\sin^2 x = \frac{1}{2}(1$

$\sum_{n=1}^{\infty}$