81. $\tan\psi = \tan(\phi - \theta) = \dfrac{\tan\phi - \tan\theta}{1 + \tan\phi\,\tan\theta} = \dfrac{\dfrac{dy}{dx} - \tan\theta}{1 + \dfrac{dy}{dx}\tan\theta} = \dfrac{\dfrac{dy/d\theta}{dx/d\theta} - \tan\theta}{1 + \dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$

$$= \frac{\dfrac{dy}{d\theta} - \dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta} + \dfrac{dy}{d\theta}\tan\theta} = \frac{\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right) - \tan\theta\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right) + \tan\theta\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right)}$$

$$= \frac{r\cos\theta + r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta + \dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}} = \frac{r\cos^2\theta + r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta + \dfrac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

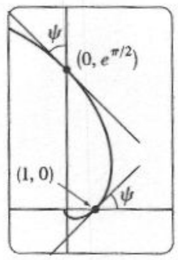
82. (a) $r = e^\theta \;\Rightarrow\; dr/d\theta = e^\theta$, so by Exercise 81, $\tan\psi = r/e^\theta = 1 \;\Rightarrow\; \psi = \arctan 1 = \frac{\pi}{4}$.

(b) The Cartesian equation of the tangent line at $(1,0)$ is $y = x - 1$, and that of the tangent line at $\left(0, e^{\pi/2}\right)$ is $y = e^{\pi/2} - x$.

(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a = \tan\psi$. Then, by Exercise 81,

$a = \dfrac{r}{dr/d\theta} \;\Rightarrow\; \dfrac{dr}{d\theta} = \dfrac{1}{a}r \;\Rightarrow$

$r = Ce^{\theta/a}$ (by Theorem 10.4.2).



11.4 Areas and Lengths in Polar Coordinates

1. $r = \sqrt{\theta},\ 0 \le \theta \le \frac{\pi}{4}$. $A = \int_0^{\pi/4} \frac{1}{2}r^2\,d\theta = \int_0^{\pi/4} \frac{1}{2}\left(\sqrt{\theta}\right)^2 d\theta = \int_0^{\pi/4} \frac{1}{2}\theta\,d\theta = \left[\frac{1}{4}\theta^2\right]_0^{\pi/4} = \frac{1}{64}\pi^2$

2. $r = e^{\theta/2},\ \pi \le \theta \le 2\pi$. $A = \int_\pi^{2\pi} \frac{1}{2}\left(e^{\theta/2}\right)^2 d\theta = \int_\pi^{2\pi} \frac{1}{2}e^\theta\,d\theta = \frac{1}{2}\left[e^\theta\right]_\pi^{2\pi} = \frac{1}{2}\left(e^{2\pi} - e^\pi\right)$

3. $r = \sin\theta,\ \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$.

$A = \int_{\pi/3}^{2\pi/3} \frac{1}{2}\sin^2\theta\,d\theta = \frac{1}{4}\int_{\pi/3}^{2\pi/3}(1 - \cos 2\theta)\,d\theta = \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{2\pi/3}$

$= \frac{1}{4}\left[\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} - \frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right] = \frac{1}{4}\left[\frac{2\pi}{3} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{\pi}{3} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{1}{4}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$

4. $r = \sqrt{\sin\theta},\ 0 \le \theta \le \pi$. $A = \int_0^\pi \frac{1}{2}\left(\sqrt{\sin\theta}\right)^2 d\theta = \int_0^\pi \frac{1}{2}\sin\theta\,d\theta = \left[-\frac{1}{2}\cos\theta\right]_0^\pi = \frac{1}{2} + \frac{1}{2} = 1$

5. $r = \theta,\ 0 \le \theta \le \pi$. $A = \int_0^\pi \frac{1}{2}\theta^2 d\theta = \left[\frac{1}{6}\theta^3\right]_0^\pi = \frac{1}{6}\pi^3$

6. $r = 1 + \sin\theta,\ \frac{\pi}{2} \le \theta \le \pi$.

$A = \int_{\pi/2}^\pi \frac{1}{2}(1 + \sin\theta)^2\,d\theta = \frac{1}{2}\int_{\pi/2}^\pi (1 + 2\sin\theta + \sin^2\theta)\,d\theta = \frac{1}{2}\int_{\pi/2}^\pi \left[1 + 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\right] d\theta$

$= \frac{1}{2}\left[\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\pi/2}^\pi = \frac{1}{2}\left[\pi + 2 + \frac{\pi}{2} - 0 - \left(\frac{\pi}{2} - 0 + \frac{\pi}{4} - 0\right)\right] = \frac{1}{2}\left(\frac{3\pi}{4} + 2\right) = \frac{3\pi}{8} + 1$

7. $r = 4 + 3\sin$

A

8. $r = \sin 4\theta, 0$

9. The area above
$\theta = 0$ to $\theta =$
$A = 2\int_0^{\pi/2} \frac{1}{2}$
$= 3^2\int_0^{\pi/2}$
$= \frac{9}{2}\left[\theta + \frac{1}{2}\right.$

Also, note tha

10. $A = \int_0^{2\pi} \frac{1}{2}r^2$
$= \frac{9}{2}\int_0^{2\pi}(1$
$= \frac{9}{2}\int_0^{2\pi}[$
$= \frac{9}{2}\left[\frac{3}{2}\theta +\right.$

11. The curve r^2
$\theta = \pi/4$, so
multiply it by
$A = 4\int_0^{\pi/4}$
$= 8\int_0^{\pi/4}$

12. The curve r^2
so we'll find
$A = 2\int_0^{\pi/2}$
$= -\frac{1}{2}(-1$

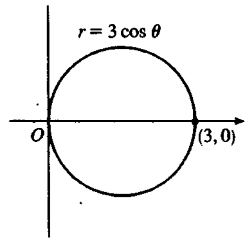
7. $r = 4 + 3\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}(4 + 3\sin\theta)^2 d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 24\sin\theta + 9\sin^2\theta)\, d\theta \\ &= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 9\sin^2\theta)\, d\theta \quad \text{[by Theorem 5.5.6(b)]} \\ &= \tfrac{1}{2}\cdot 2\int_0^{\pi/2} \left[16 + 9\cdot\tfrac{1}{2}(1 - \cos 2\theta)\right] d\theta \quad \text{[by Theorem 5.5.6(a)]} \\ &= \int_0^{\pi/2} \left(\tfrac{41}{2} - \tfrac{9}{2}\cos 2\theta\right) d\theta = \left[\tfrac{41}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \left(\tfrac{41\pi}{4} - 0\right) - (0 - 0) = \tfrac{41\pi}{4} \end{aligned}$$

8. $r = \sin 4\theta,\ 0 \le \theta \le \frac{\pi}{4}$. $A = \int_0^{\pi/4} \frac{1}{2}\sin^2 4\theta\, d\theta = \int_0^{\pi/4} \frac{1}{4}(1 - \cos 8\theta)\, d\theta = \left[\frac{1}{4}\theta - \frac{1}{32}\sin 8\theta\right]_0^{\pi/4} = \frac{\pi}{16}$

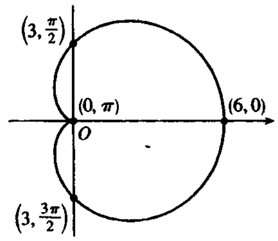
9. The area above the polar axis is bounded by $r = 3\cos\theta$ for $\theta = 0$ to $\theta = \pi/2$ (*not* π). By symmetry,

$$\begin{aligned} A &= 2\int_0^{\pi/2} \tfrac{1}{2}r^2\, d\theta = \int_0^{\pi/2} (3\cos\theta)^2 d\theta \\ &= 3^2\int_0^{\pi/2} \cos^2\theta\, d\theta = 9\int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta \\ &= \tfrac{9}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{9}{2}\left[\left(\tfrac{\pi}{2} + 0\right) - (0 + 0)\right] = \tfrac{9\pi}{4}. \end{aligned}$$



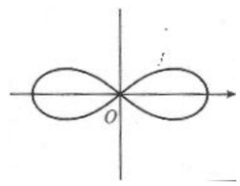
Also, note that this is a circle with radius $\frac{3}{2}$, so its area is $\pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$.

10. $$\begin{aligned} A &= \int_0^{2\pi} \tfrac{1}{2}r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}\left[3(1 + \cos\theta)\right]^2 d\theta \\ &= \tfrac{9}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta \\ &= \tfrac{9}{2}\int_0^{2\pi} \left[1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\ &= \tfrac{9}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{27}{2}\pi \end{aligned}$$



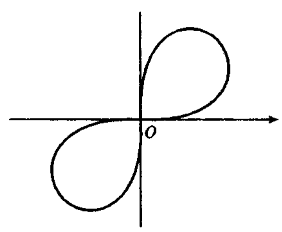
11. The curve $r^2 = 4\cos 2\theta$ goes through the pole when $\theta = \pi/4$, so we'll find the area for $0 \le \theta \le \pi/4$ and multiply it by 4.

$$\begin{aligned} A &= 4\int_0^{\pi/4} \tfrac{1}{2}r^2\, d\theta = 2\int_0^{\pi/4} (4\cos 2\theta)\, d\theta \\ &= 8\int_0^{\pi/4} \cos 2\theta\, d\theta = 4\left[\sin 2\theta\right]_0^{\pi/4} = 4(1 - 0) = 4 \end{aligned}$$



12. The curve $r^2 = \sin 2\theta$ goes through the pole when $\theta = \pi/2$, so we'll find the area for $0 \le \theta \le \pi/2$ and multiply it by 2.

$$\begin{aligned} A &= 2\int_0^{\pi/2} \tfrac{1}{2}r^2\, d\theta = \int_0^{\pi/2} \sin 2\theta\, d\theta = -\tfrac{1}{2}\left[\cos 2\theta\right]_0^{\pi/2} \\ &= -\tfrac{1}{2}(-1 - 1) = -\tfrac{1}{2}(-2) = 1 \end{aligned}$$

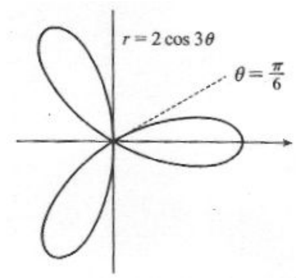


13. One-sixth of the area lies above the polar axis and is bounded by the curve $r = 2\cos 3\theta$ for $\theta = 0$ to $\theta = \pi/6$.

$A = 6\int_0^{\pi/6} \frac{1}{2}(2\cos 3\theta)^2\,d\theta = 12\int_0^{\pi/6} \cos^2 3\theta\,d\theta$

$= \frac{12}{2}\int_0^{\pi/6}(1 + \cos 6\theta)d\theta$

$= 6\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 6\left(\frac{\pi}{6}\right) = \pi$

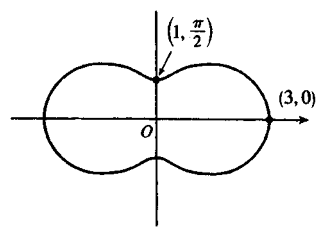


14. $A = \int_0^{2\pi} \frac{1}{2}(2 + \cos 2\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4 + 4\cos 2\theta + \cos^2 2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left(4 + 4\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right)\,d\theta$

$= \frac{1}{2}\left[\frac{9}{2}\theta + 2\sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{2\pi}$

$= \frac{1}{2}(9\pi) = \frac{9\pi}{2}$



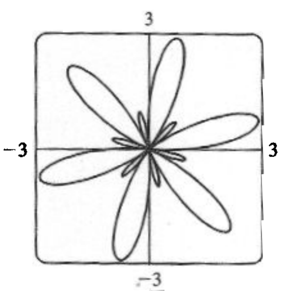
15. $A = \int_0^{2\pi} \frac{1}{2}(1 + 2\sin 6\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1 + 4\sin 6\theta + 4\sin^2 6\theta)d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[1 + 4\sin 6\theta + 4\cdot\frac{1}{2}(1 - \cos 12\theta)\right]\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}(3 + 4\sin 6\theta - 2\cos 12\theta)\,d\theta$

$= \frac{1}{2}\left[3\theta - \frac{2}{3}\cos 6\theta - \frac{1}{6}\sin 12\theta\right]_0^{2\pi}$

$= \frac{1}{2}\left[\left(6\pi - \frac{2}{3} - 0\right) - \left(0 - \frac{2}{3} - 0\right)\right] = 3\pi.$



16. $A = \int_0^{\pi} \frac{1}{2}(2\sin\theta + 3\sin 9\theta)^2\,d\theta = 2\int_0^{\pi/2}\frac{1}{2}(2\sin\theta + 3\sin 9\theta)^2\,d\theta$

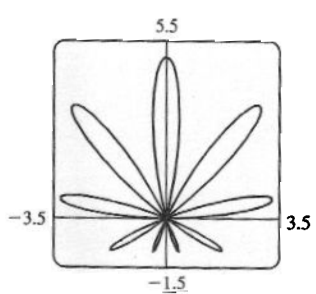
$= \int_0^{\pi/2}(4\sin^2\theta + 12\sin\theta\sin 9\theta + 9\sin^2 9\theta)\,d\theta$

$= \int_0^{\pi/2}\left[2(1 - \cos 2\theta) + 12\cdot\frac{1}{2}(\cos(\theta - 9\theta) - \cos(\theta + 9\theta)) + \frac{9}{2}(1 - \cos 18\theta)\right]\,d\theta$

[integration by parts could be used for $\int \sin\theta\sin 9\theta\,d\theta$]

$= \int_0^{\pi/2}\left(2 - 2\cos 2\theta + 6\cos 8\theta - 6\cos 10\theta + \frac{9}{2} - \frac{9}{2}\cos 18\theta\right)\,d\theta$

$= \left[\frac{13}{2}\theta - \sin 2\theta + \frac{3}{4}\sin 8\theta - \frac{3}{5}\sin 10\theta - \frac{1}{4}\sin 18\theta\right]_0^{\pi/2} = \frac{13}{4}\pi$



17. The shaded loo

to $\theta = \pi/2$.

$A = \int_0^{\pi/2}$

$= \frac{1}{2}\int_0^{\pi}$

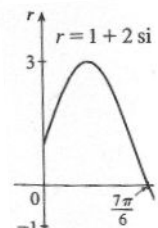
$= \frac{1}{4}[\theta$ -

19. $r = 0 \;\Rightarrow\; 3$

$A = \int_{-\pi/10}^{\pi/10}\frac{1}{2}($

20. $A = 2\int_0^{\pi/8}\frac{1}{2}($

21.



$A =$

$=$

$=$

22. To determine w

through the pol

$2\cos^2\theta - 1 =$

$\theta = \frac{\pi}{4}$ or $\theta =$

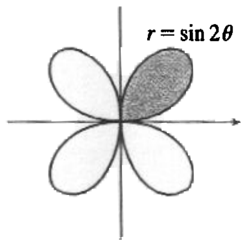
$A = 2\int_0^{\pi/4}\frac{1}{2}($

$= \int_0^{\pi/4}[4\cdot$

$= [-2\theta + \sin$

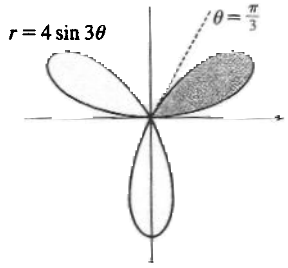
17. The shaded loop is traced out from $\theta = 0$ to $\theta = \pi/2$.

$$A = \int_0^{\pi/2} \tfrac{1}{2} r^2\, d\theta = \tfrac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\, d\theta$$
$$= \tfrac{1}{2}\int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 4\theta)\, d\theta$$
$$= \tfrac{1}{4}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \tfrac{1}{4}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{8}$$



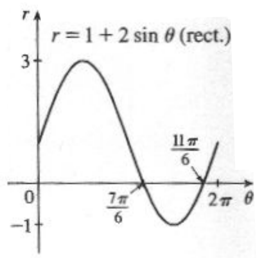
18. $A = \int_0^{\pi/3} \tfrac{1}{2}(4\sin 3\theta)^2\, d\theta = 8\int_0^{\pi/3} \sin^2 3\theta\, d\theta$

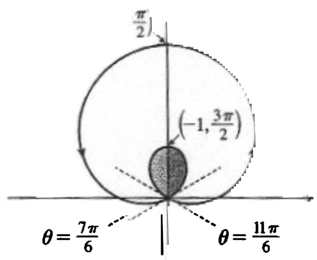
$$= 4\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta$$
$$= 4\left[\theta - \tfrac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \tfrac{4\pi}{3}$$



19. $r = 0 \;\Rightarrow\; 3\cos 5\theta = 0 \;\Rightarrow\; 5\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{10}$.

$$A = \int_{-\pi/10}^{\pi/10} \tfrac{1}{2}(3\cos 5\theta)^2\, d\theta = \int_0^{\pi/10} 9\cos^2 5\theta\, d\theta = \tfrac{9}{2}\int_0^{\pi/10}(1 + \cos 10\theta)\, d\theta = \tfrac{9}{2}\left[\theta + \tfrac{1}{10}\sin 10\theta\right]_0^{\pi/10} = \tfrac{9\pi}{20}$$





This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

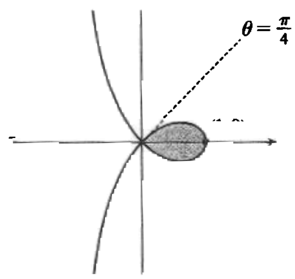
$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}(1 + 2\sin\theta)^2\, d\theta = \int_{7\pi/6}^{3\pi/2}\left(1 + 4\sin\theta + 4\sin^2\theta\right) d\theta$$
$$= \int_{7\pi/6}^{3\pi/2}\left[1 + 4\sin\theta + 4\cdot\tfrac{1}{2}(1 - \cos 2\theta)\right] d\theta = \left[\theta - 4\cos\theta + 2\theta - \sin 2\theta\right]_{7\pi/6}^{3\pi/2}$$
$$= \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

n $9\theta\, d\theta]$

22. To determine when the strophoid $r = 2\cos\theta - \sec\theta$ passes through the pole, we solve $r = 0 \;\Rightarrow\; 2\cos\theta - \dfrac{1}{\cos\theta} = 0 \;\Rightarrow$

$2\cos^2\theta - 1 = 0 \;\Rightarrow\; \cos^2\theta = \dfrac{1}{2} \;\Rightarrow\; \cos\theta = \pm\dfrac{1}{\sqrt{2}} \;\Rightarrow$

$\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ for $0 \le \theta \le \pi$ with $\theta \ne \frac{\pi}{2}$.



$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(2\cos\theta - \sec\theta)^2\, d\theta = \int_0^{\pi/4}(4\cos^2\theta - 4 + \sec^2\theta)\, d\theta$$
$$= \int_0^{\pi/4}\left[4\cdot\tfrac{1}{2}(1 + \cos 2\theta) - 4 + \sec^2\theta\right] d\theta = \int_0^{\pi/4}(-2 + 2\cos 2\theta + \sec^2\theta)\, d\theta$$
$$= \left[-2\theta + \sin 2\theta + \tan\theta\right]_0^{\pi/4} = \left(-\tfrac{\pi}{2} + 1 + 1\right) - 0 = 2 - \tfrac{\pi}{2}$$