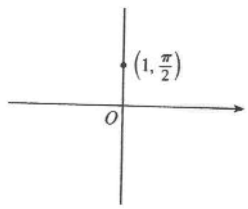
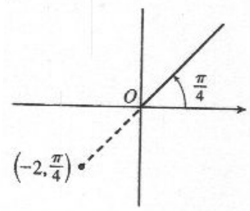
11.3 Polar Coordinates

1. (a) By adding 2π to $\frac{\pi}{2}$, we obtain the point $\left(1, \frac{5\pi}{2}\right)$. The direction opposite $\frac{\pi}{2}$ is $\frac{3\pi}{2}$, so $\left(-1, \frac{3\pi}{2}\right)$ is a point that satisfies the $r < 0$ requirement.

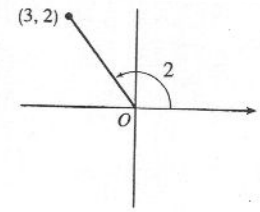


(b) $\left(-2, \frac{\pi}{4}\right)$



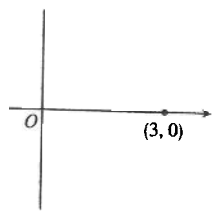
$\left(2, \frac{5\pi}{4}\right), \left(-2, \frac{9\pi}{4}\right)$

(c) $(3, 2)$



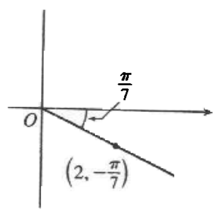
$(3, 2 + 2\pi), (-3, 2 + \pi)$

2. (a) $(3, 0)$



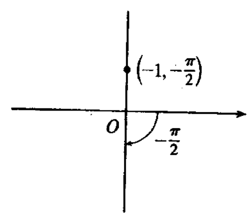
$(3, 2\pi), (-3, \pi)$

(b) $\left(2, -\frac{\pi}{7}\right)$



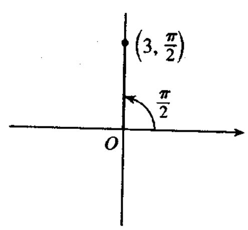
$\left(2, \frac{13\pi}{7}\right), \left(-2, \frac{6\pi}{7}\right)$

(c) $\left(-1, -\frac{\pi}{2}\right)$



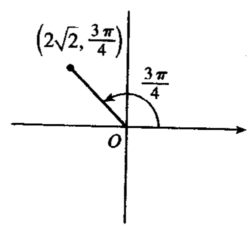
$\left(1, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right)$

3. (a)



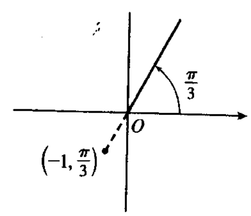
$x = 3\cos\frac{\pi}{2} = 3(0) = 0$ and $y = 3\sin\frac{\pi}{2} = 3(1) = 3$ give us the Cartesian coordinates $(0, 3)$.

(b)



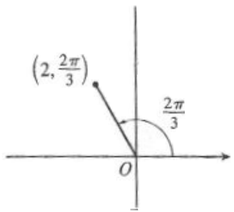
$x = 2\sqrt{2}\cos\frac{3\pi}{4} = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -2$ and $y = 2\sqrt{2}\sin\frac{3\pi}{4} = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$ give us $(-2, 2)$.

(c)



$x = -1\cos\frac{\pi}{3} = -\frac{1}{2}$ and $y = -1\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$ give us $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

4. (a)

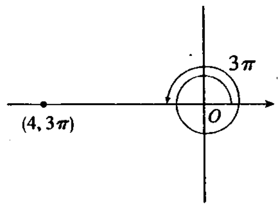


$x = 2\cos\frac{2\pi}{3} = -1$ and

$y = 2\sin\frac{2\pi}{3} = \sqrt{3}$ give

us $(-1, \sqrt{3})$.

(b)

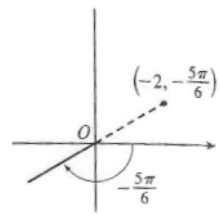


$x = 4\cos 3\pi = -4$ and

$y = 4\sin 3\pi = 0$ give

us $(-4, 0)$.

(c)



$x = -2\cos(-\frac{5\pi}{6}) = \sqrt{3}$

and $y = -2\sin(-\frac{5\pi}{6}) = 1$

give us $(\sqrt{3}, 1)$.

5. (a) $x = 1$ and $y = 1$ $\Rightarrow$ $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$. Since $(1, 1)$ is in the first quadrant, the polar coordinates are (i) $(\sqrt{2}, \frac{\pi}{4})$ and (ii) $(-\sqrt{2}, \frac{5\pi}{4})$.

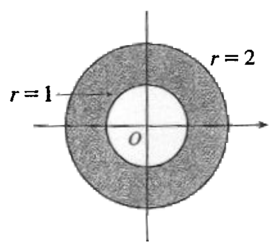
(b) $x = 2\sqrt{3}$ and $y = -2$ $\Rightarrow$ $r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$ and

$\theta = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$. Since $(2\sqrt{3}, -2)$ is in the fourth quadrant and $0 \leq \theta \leq 2\pi$, the polar coordinates are (i) $(4, \frac{11\pi}{6})$ and (ii) $(-4, \frac{5\pi}{6})$.

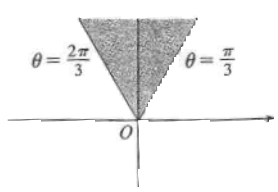
6. (a) $(x, y) = (-1, -\sqrt{3})$, $r = \sqrt{1 + 3} = 2$, $\tan\theta = y/x = \sqrt{3}$ and (x, y) is in the third quadrant, so $\theta = \frac{4\pi}{3}$. The polar coordinates are (i) $(2, \frac{4\pi}{3})$ and (ii) $(-2, \frac{\pi}{3})$.

(b) $(x, y) = (-2, 3)$, $r = \sqrt{4 + 9} = \sqrt{13}$, $\tan\theta = y/x = -\frac{3}{2}$ and (x, y) is in the second quadrant, so $\theta = \tan^{-1}(-\frac{3}{2}) + \pi$. The polar coordinates are (i) $(\sqrt{13}, \theta)$ and (ii) $(-\sqrt{13}, \theta + \pi)$.

7. The curves $r = 1$ and $r = 2$ represent circles with center O and radii 1 and 2. The region in the plane satisfying $1 \leq r \leq 2$ consists of both circles and the shaded region between them in the figure.



8. $r \geq 0$, $\pi/3 \leq \theta \leq 2\pi/3$



9. The region sa
$-\pi/2 \leq \theta <$
$r = 4$ nor the

11. $2 < r < 3$,

13. $(r, \theta) = (1, \frac{\pi}{6})$
$(r, \theta) = (3, \frac{3\pi}{4}$

14. The points $(r_1,$
respectively. Th

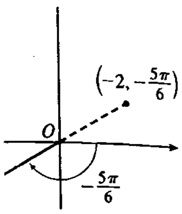
$(r_2\cos\theta_2 - r$
$= (r_2^2$
$= r_1^2($
$= r_1^2 -$

so the distance b

15. $r = 2$ $\Leftrightarrow$

16. $r\cos\theta = 1$ $\Leftrightarrow$

17. $r = 3\sin\theta$ $\Rightarrow$
at $(0, \frac{3}{2})$. The fir
or $r = 3\sin\theta$. B
equivalent to the



$-2\cos(-\frac{5\pi}{6}) = \sqrt{3}$

$= -2\sin(-\frac{5\pi}{6}) = 1$

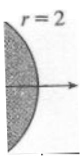
s $(\sqrt{3}, 1)$.

the first quadrant, the

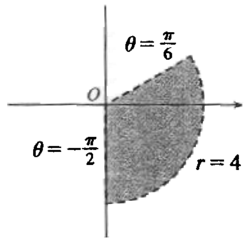
and $0 \le \theta \le 2\pi$, the

drant, so $\theta = \frac{4\pi}{3}$.

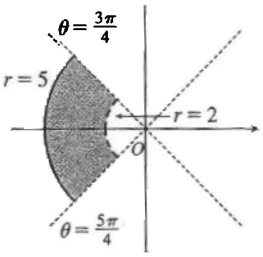
ıadrant, so



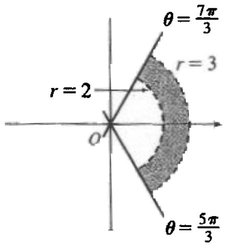
9. The region satisfying $0 \le r < 4$ and $-\pi/2 \le \theta < \pi/6$ does not include the circle $r = 4$ nor the line $\theta = \frac{\pi}{6}$.



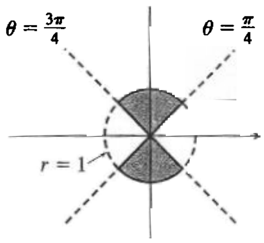
10. $2 < r \le 5, \quad 3\pi/4 < \theta < 5\pi/4$



11. $2 < r < 3, \quad \frac{5\pi}{3} \le \theta \le \frac{7\pi}{3}$



12. $-1 \le r \le 1, \quad \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$



13. $(r, \theta) = \left(1, \frac{\pi}{6}\right) \Rightarrow x = 1\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $y = 1\sin\frac{\pi}{6} = \frac{1}{2}$.

$(r, \theta) = \left(3, \frac{3\pi}{4}\right) \Rightarrow x = 3\cos\frac{3\pi}{4} = -\frac{3\sqrt{2}}{2}$ and $y = 3\sin\frac{3\pi}{4} = \frac{3\sqrt{2}}{2}$. The distance between them is

$$\sqrt{\left[\frac{\sqrt{3}}{2} - \left(-\frac{3\sqrt{2}}{2}\right)\right]^2 + \left(\frac{1}{2} - \frac{3\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{4}\left(\sqrt{3} + 3\sqrt{2}\right)^2 + \frac{1}{4}\left(1 - 3\sqrt{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}}\sqrt{\left(3 + 6\sqrt{6} + 18\right) + \left(1 - 6\sqrt{2} + 18\right)} = \frac{1}{2}\sqrt{40 + 6\sqrt{6} - 6\sqrt{2}}$$

14. The points (r_1, θ_1) and (r_2, θ_2) in Cartesian coordinates are $(r_1\cos\theta_1, r_1\sin\theta_1)$ and $(r_2\cos\theta_2, r_2\sin\theta_2)$, respectively. The *square* of the distance between them is

$$\begin{aligned}
&(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2 \\
&\quad = \left(r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1\right) + \left(r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1\right) \\
&\quad = r_1^2\left(\sin^2\theta_1 + \cos^2\theta_1\right) + r_2^2\left(\sin^2\theta_2 + \cos^2\theta_2\right) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\
&\quad = r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2,
\end{aligned}$$

so the distance between them is $\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r = 2 \Leftrightarrow \sqrt{x^2 + y^2} = 2 \Leftrightarrow x^2 + y^2 = 4$, a circle of radius 2 centered at the origin.

16. $r\cos\theta = 1 \Leftrightarrow x = 1$, a vertical line.

17. $r = 3\sin\theta \Rightarrow r^2 = 3r\sin\theta \Leftrightarrow x^2 + y^2 = 3y \Leftrightarrow x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$, a circle of radius $\frac{3}{2}$ centered at $\left(0, \frac{3}{2}\right)$. The first two equations are actually equivalent since $r^2 = 3r\sin\theta \Rightarrow r(r - 3\sin\theta) = 0 \Rightarrow r = 0$ or $r = 3\sin\theta$. But $r = 3\sin\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the single equation $r = 3\sin\theta$ is equivalent to the compound condition ($r = 0$ or $r = 3\sin\theta$).