SECTION 12.8: Power Series

Introduction:

So far we have only talked about series of constants. For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}.$$

We asked if such series converge or diverge.

Now, we want to discuss series of *functions*. For example,

$$\sum_{n=1}^{\infty} \frac{\sin nx}{2^n}.$$

Now the question we ask is:

For what values of x does the series converge?

Another related question is:

If the series converges for some values of x, what function does it converge to; that is, what is $s(x)$?

We are only going to discuss very special series of function called power series.

Power Series and Convergence:

A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a power series centered at $x = a$. Each partial sum is a polynomial.

Example:

$$\sum_{n=0}^{\infty} \frac{(2x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n(x+\frac{1}{2})^n}{n!}$$

$$= 1 + 2(x+\tfrac{1}{2}) + \frac{2^2(x+\frac{1}{2})^2}{2} + \frac{2^3(x+\frac{1}{2})^3}{3!} + \frac{2^4(x+\frac{1}{2})^4}{4!} + \cdots$$

The *coefficients* are: $c_0 = 1$, $c_1 = 2$, $c_2 = 2$, $c_3 = \frac{4}{3}$, $c_4 = \frac{2}{3}$.

The *center* is: $a = -\frac{1}{2}$.

The infinite power series is a function of x defined for those values of x for which the series converges. The set of values of x for which the power series converges is called the *interval of convergence* or the *convergence set*.

The interval of convergence, $a - r < x < a + r$, has a *radius of convergence*, r.

For the power series,

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n,$$

we use the Ratio Test to determine the domain of $f(x)$ and thus determines the convergence set.

$$\lim_{n \to \infty} \left| \frac{c_{n+1}(x - a)^{n+1}}{c_n(x - a)^n} \right| = \lim_{n \to \infty} \left| \frac{c_{n+1}(x - a)}{c_n} \right| = |x - a| \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|$$

Notice the Ratio Test tells us that

$$|x - a| \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|$$

must be less than 1 for the series to converge. Let

$$\rho = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|$$

and consider the following three cases.

CASE 1 ($\rho = 0$): If $\rho = 0$, then the power series is convergence for all x since

$$|x - a|\, \rho = 0 < 1.$$

The interval of convergence is $-\infty < x < \infty$ or $(-\infty, \infty)$ and the radius of convergence is $r = \infty$.

Example: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n + 1)!} \cdot \frac{n!}{x^n} \right| = |x| \lim_{n \to \infty} \frac{1}{n + 1} = 0 \text{ for all values of } x$$

center: $a = 0$
radius of convergence: $r = \infty$
convergence set: $(-\infty, \infty)$

CASE 2 ($\rho = \infty$): If $\rho = \infty$, then the power series converges for $x = a$ only since by the ratio test the series diverges for all values of x except $x = a$ ($r = 0$, $x = a$).

Example: $\sum_{n=0}^{\infty} n!(x - 2)^n$

$$\lim_{n \to \infty} \left| \frac{(n + 1)!(x - 2)^{n+1}}{n!\,(x - 2)^n} \right| = |x - 2| \lim_{n \to \infty} (n + 1) = \infty$$

$$\text{for all values of } x \text{ except } a = 2$$

center: $a = 2$
radius of convergence: $r = 0$
convergence set: $x = 2$

CASE 3 ($\rho \neq 0$ or ∞): If $\rho \neq 0$ and $\rho \neq \infty$, then by the Ratio Test,

$$|x - a|\rho$$

must be less than 1 for the series to converge. The series, not counting endpoints, *converges absolutely* for those values of x such that

$$|x - a|\rho < 1 \quad \text{or} \quad |x - a| < \frac{1}{\rho}.$$

So, $\frac{1}{\rho}$ is the radius of convergence.

To determine whether the endpoints are included in the domain (interval of convergence), a test *other than the ratio test* must be used. (Recall when $|x - a|\rho = 1$ to ratio test fails.)

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{n(2x + 1)^n}{n + 1} = \sum_{n=1}^{\infty} \frac{n2^n(x + ½)^n}{n + 1}$

$$\lim_{n \to \infty} \left| \frac{(n + 1)2^{n+1}(x + ½)^{n+1}}{n + 2} \cdot \frac{n + 1}{n2^n(x + ½)^n} \right| = |x + ½| \lim_{n \to \infty} \frac{2(n + 1)^2}{n(n + 2)}$$

$$= |x + ½| \lim_{n \to \infty} \frac{2n^2 + 4n + 2}{n^2 + 2n}$$

$$= |x + ½|2$$

Thus, the series converges when

$$|x + ½| \cdot 2 < 1 \quad \text{or} \quad |x + ½| < ½$$

Hence,

$$x + ½ < ½ \qquad -(x + ½) < ½$$

$$x < 0 \qquad x + ½ > -½$$

$$x > -1$$

center: $a = -½$

radius of convergence: $r = ½$

Now before stating the interval of convergence, we need to check the endpoints of the interval, namely $x = -1$ and $x = 0$.

When $x = -1$, the power series is

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{n + 1}.$$

This series diverges by the n^{th} term test since

$$\lim_{n \to \infty} \frac{n}{n + 1} = 1.$$

When $x = 0$, the power series is

$$\sum_{n=1}^{\infty} \frac{n}{n + 1}$$

which also diverges by the n^{th} term test since

$$\lim_{n \to \infty} \frac{n}{n + 1} = 1.$$

Thus, the interval of convergence is

$$-1 < x < 0 \text{ or } (-1, 0),$$

the radius of convergence is

$$r = ½,$$

and the center is

$$a = -½.$$

ADDITIONAL EXAMPLES: Find the interval and radius of convergence of the following power series.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^{2n}}{n^3}$