# <u>Section 12.2</u>: Series

**<u>Definitions</u>:**

Let $\{a_n\}$ be a sequence of real numbers, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

is the ***<u>infinite series</u>*** (or just a ***<u>series</u>***) associated with the sequence.

Its ***<u>partial sums</u>*** are:

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

If $\lim\limits_{n \to \infty} s_n = s$ exists and is finite, then $S$ is the <u>sum</u> of the infinite series and

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = s .$$

If $S$ exists and is finite, the series ***<u>converges</u>***, otherwise the series ***<u>diverges</u>***.

NOTES:

1. A <u>sequence</u> is a listing of numbers, $\{a_1, a_2, a_3, \ldots\}$; a <u>series</u> is a sum of numbers, $a_1 + a_2 + a_3 + \ldots$.
2. Every series involves <u>two</u> sequences:
   (a) a sequence of terms, $a_1, a_2, a_3, \ldots$, and
   (b) a sequence of partial sums, $s_1, s_2, s_3, \ldots$.

Example: Let $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\right\}$ be a <u>sequence</u> where $a_n = \dfrac{1}{2^{n-1}}$. The <u>infinite series</u>,

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

has <u>partial sums</u>:

$$s_1 = 1$$
$$s_2 = 1 + \tfrac{1}{2} = \tfrac{3}{2}$$
$$s_3 = 1 + \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{7}{4}$$
$$\vdots$$
$$s_n = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$

(A later subsection will show to determine $s_n$.)
In summary,

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \lim_{n \to \infty} \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2\,.$$

**Graphing the Partial Sums of a Series on the TI-83/84:**

1. Press **Y=.**
2. In one of the sequence variables, enter the following.
3. Press **2nd, STAT**. Arrow over to **MATH**. Select **5:sum(**.
4. Press **2nd, STAT**. Arrow over to **OPS**. Select **5:seq(**.
5. Enter the formula for the sequence of terms; that is, the $a_n$.
6. Press the comma (**,**), ***n***, the starting index, ***n***.
7. Finish by closing both sets of parentheses.

Examples: Graph both the sequence of terms and the sequence of partial sums for the following series. Decide based upon your graphs of the partial sums if you think the series converges or diverges

1. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{2^n}\right)$

2. $\sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)$

3. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)$

**Geometric Series:**

$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots \quad (a \neq 0)$ is called a ***geometric series*** with ratio $r$.

Let's determine when a geometric series converges. We do this by considering $s_n$ and $rs_n$.

$$\begin{aligned} s_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ rs_n &= \quad\;\; ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

First, we note that if $r = 1$, $s_n = na$, which grows without bound, and so $\{s_n\}$ diverges. Now, for $r \neq 1$, we subtract the second equation above from the first and get

$$\begin{aligned} s_n - rs_n &= a - ar^n \\ s_n(1 - r) &= a - ar^n \\ s_n &= \frac{a - ar^n}{1 - r}. \end{aligned}$$

If $|r| < 1$, we know from the last section that $\lim_{n \to \infty} r^n = 0$ and thus

$$s = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a - ar^n}{1 - r} = \frac{a}{1 - r}.$$

If $|r| > 1$ or $r = -1$, the sequence $\{r^n\}$ diverges, and consequently so does $\{S_n\}$.

So, we have the following theorem.

**Theorem:** A geometric series, $\sum_{n=1}^{\infty} ar^{n-1} \left( \text{or } \sum_{n=0}^{\infty} ar^n \right)$, converges to $s = \frac{a}{1 - r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Examples: Determine whether the following series converge or diverge. If the series converges, find its sum.

1. $\sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^{n-1}$

2. $\sum_{n=1}^{\infty} \left( -\frac{1}{3} \right)^{n-1}$

3. $\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}}$

**Telescoping Series:**

A ***telescoping series*** is one in which each partial sum collapses (or telescopes). Sometimes, telescoping series are also called ***collapsing series***. See the example below.

Example: Show the following series converges and find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

**<u>A Test for Divergence</u>:**

**The $n^{th}$ Term Test:** If $\lim_{n \to \infty} a_n \neq 0$, then $\Sigma a_n$ <u>diverges</u>.

Examples: Use the $n^{th}$ term test to show the following series diverge.

1. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

2. $\sum_{n=1}^{\infty} (-1)^n$

NOTES:

1. If $\lim_{n \to \infty} a_n = 0$, the series may <u>converge</u> or <u>diverge</u>. (See the next subsection <u>The Harmonic Series</u>.)

2. One can *only conclude <u>divergence</u>* with the $n^{th}$ term test!

3. It is <u>necessary</u> that $\lim_{n \to \infty} a_n = 0$ for $\Sigma a_n$ to converge, but it is not sufficient to <u>conclude</u> convergence!

Proof of the $n^{th}$ Term Test:

We will actually prove what is known as the contrapositive of the $n^{th}$ Term Test. The contrapositive of the $n^{th}$ Term Test is:

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$. (1)

Logically, the statement above is equivalent to the original statement of the $n^{\text{th}}$ Term Test.

Now, we proceed to prove statement (1) above. We assume that $\sum_{n=1}^{\infty} a_n$ converges; that is,

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = L.$$

Since $s_n = s_{n-1} + a_n$ and $\lim_{n \to \infty} s_n = \lim_{n \to \infty} s_{n-1} = L$, then :

$$L = \lim_{n \to \infty} s_n = \lim_{n \to \infty} (s_{n-1} + a_n)$$

$$L = \lim_{n \to \infty} s_{n-1} + \lim_{n \to \infty} a_n$$

$$L = L + \lim_{n \to \infty} a_n.$$

Therefore, $\lim_{n \to \infty} a_n = 0$ . By the contrapositive we have shown that:

If $\lim_{n \to \infty} a_n \neq 0$, then $\Sigma a_n$ diverges.

**<u>The Harmonic Series</u>:**

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is called the <u>*harmonic*</u> <u>*series*</u>.

We note that

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

However, the harmonic series diverges as we will now show.

What we will show is that the partial sums $s_n$ of the harmonic series grow with out bound, that is approaches infinity. Suppose that $n$ is large. The $n^{\text{th}}$ partial sum is

$$
\begin{aligned}
s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&\quad + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots + \frac{1}{n} \\
&> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \cdots + \frac{1}{n} \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{n}
\end{aligned}
$$

By taking $n$ large enough, we can introduce as many ½'s into the last line as we wish. Thus, we see that $s_n$ can be made larger than any number we want; that is, $s_n$ increases without bound (approaches infinity). Hence, $\{s_n\}$ diverges which tells us the harmonic series diverges.

**Theorem:** If $\Sigma a_n$ and $\Sigma b_n$ are convergent series , then so are the series $\Sigma ca_n$ (where $c$ is a constant), $\Sigma(a_n + b_n)$, and $\Sigma(a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example: Find the sum of the series $\sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} - \frac{1}{3^n} \right)$

**Theorem:** If $\sum_{n=1}^{\infty} a_n$ diverges and $c \neq 0$, then $\sum_{n=1}^{\infty} ca_n$ also diverges.

Example: Show that $\sum_{n=1}^{\infty} \frac{1000}{n}$ diverges.