Section 12.10: Taylor and Maclaurin Series

The Uniqueness Theorem: Suppose f satisfies

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots$$

for all x in some interval around a. Then

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Thus, a function cannot have more than one power series in $x - a$ that represents it.

NOTES:

1. Consider $f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$ and the following derivatives:

$$\begin{aligned}
f(x) &= c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \cdots \\
f'(x) &= c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4a_4(x - a)^3 + \cdots \\
f''(x) &= 2c_2 + 3\cdot 2c_3(x - a) + 4\cdot 3c_4(x - a)^2 + \cdots \\
f'''(x) &= 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x - a) + \cdots \\
f^{(4)}(x) &= 4\cdot 3\cdot 2c_4 + \cdots
\end{aligned}$$

In each case, if $x = a$

$$\begin{aligned}
f(a) &= c_0 \\
f'(a) &= c_1 \\
f''(a) &= 2!c_2 \\
f'''(a) &= 3!c_3 \\
f^{(4)}(a) &= 4!c_4 \\
&\vdots \\
f^{(n)}(a) &= n!c_n
\end{aligned}$$

Solving for each coefficient:

$$c_0 = \frac{f(a)}{0!} \qquad c_1 = \frac{f'(a)}{1!}$$

$$c_2 = \frac{f''(a)}{2!} \qquad c_3 = \frac{f'''(a)}{3!}$$

$$c_4 = \frac{f^{(4)}(a)}{4!} \quad \text{and}$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

2. A power series $f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$ where coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

is called a *Taylor series*.
3. When $a = 0$, the series is known as a *Maclaurin Series*.
4. The last part of the Uniqueness Theorem tells us that there is only one power series representation for a function. In particular, the geometric series and differentiation/integration techniques of the last section yield Taylor and Maclaurin series.

Examples:

1. Find the Maclaurin series for $f(x) = \sin x$.

$$\begin{aligned} f(x) &= \sin x & f(0) &= \sin 0 = 0 \\ f'(x) &= \cos x & f'(0) &= \cos 0 = 1 \\ f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0 \\ f'''(x) &= -\cos x & f'''(0) &= -\cos 0 = -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0 \\ &\vdots & &\vdots \end{aligned}$$

So,

$$\begin{aligned} f(x) &= \sin x \\ &= f(0) + \tfrac{f'(0)}{1!}(x - 0) + \tfrac{f''(0)}{2!}(x - 0)^2 + \tfrac{f'''(0)}{3!}(x - 0)^3 + \tfrac{f^{(4)}(0)}{4!}(x - 0)^4 + \cdots \\ &= 0 + x + 0 + \tfrac{(-1)}{3!}x^3 + 0 + \tfrac{1}{5!}x^5 + 0 + \tfrac{(-1)}{7!}x^7 + \cdots \\ &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

Now, we use the Ratio Test to find the domain:

$$\lim_{n \to \infty} \left| \frac{x^{2(n+1)+1}}{[2(n+1)+1]!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = \lim_{n \to \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x^{2n+1}x^2}{(2n+3)(2n+2)(2n+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right|$$

$$= |x^2| \lim_{n \to \infty} \frac{1}{(2n+3)(2n+2)}$$

Note that $\lim_{n \to \infty} \frac{1}{(2n+3)(2n+2)} = 0$. Thus, $\rho = 0$ which implies the radius of convergence is $r = \infty$. Hence, the domain is all reals. So,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for} \quad -\infty < x < \infty.$$

2. Find the Maclaurin series for $f(x) = \cos x$.

Now, we could use the same process as we did in Example 1. However, it is easier if we recognize that $\cos x = \frac{d}{dx}(\sin x)$. Thus,

$$\begin{aligned}\cos x &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)\cdot(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\end{aligned}$$

Since the domain for $\sin x$ is all reals, the domain for $\cos x$ is also all reals; *i.e.*, $-\infty < x < \infty$ or $(-\infty, \infty)$.

3. Find the Maclaurin series for $g(x) = e^{x^2}$.

Let $f(x) = e^x$ so that $g(x) = f(x^2)$.

$$\begin{aligned}f(x) &= e^x & f(0) &= e^0 = 1 \\ f'(x) &= e^x & f'(0) &= e^0 = 1 \\ f''(x) &= e^x & f''(0) &= e^0 = 1 \\ &\vdots & &\vdots\end{aligned}$$

Thus,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and

$$\begin{aligned}e^{x^2} = g(x) = f(x^2) &= 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.\end{aligned}$$

So,

$$\left.\begin{array}{l} f(x) = e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x \\ \qquad\qquad \text{and} \\ g(x) = e^{x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \quad \text{for all } x \end{array}\right\} \text{Verify domain using Ratio Test!}$$

4. Find the Taylor Series for $f(x) = \ln x$ at $a = 1$.

Method 1: (Geometric Series)

$$f(x) = \ln x$$

$$f'(x) = \frac{1}{x} = \frac{1}{1 - (1 - x)} = \frac{1}{1 + (x - 1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n (x - 1)^n \quad \text{for } |x - 1| < 1 \text{ or } 0 < x < 2$$

So, $\displaystyle\int \sum_{n=0}^{\infty} (-1)^n (x - 1)^n = C + \sum_{n=0}^{\infty} \frac{(-1)^n (x - 1)^{n+1}}{n + 1}$.

Hence, $\displaystyle\ln x = C + \sum_{n=0}^{\infty} \frac{(-1)^n (x - 1)^{n+1}}{n + 1}$. Substituting 1 for x, we find that $C = 0$. Thus,

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x - 1)^{n+1}}{n + 1} \quad \text{for } 0 < x \le 2.$$

Verify the convergence and divergence at the endpoints.

Method 2: (Taylor's Theorem)

$$\begin{aligned}
f(x) &= \ln x & f(1) &= 0 \\
f'(x) &= x^{-1} & f'(1) &= 1 \\
f''(x) &= -x^{-2} & f''(1) &= -1 \\
f'''(x) &= 2x^{-3} & f'''(1) &= 2 \\
f^{(4)}(x) &= -6x^{-4} & f^{(4)}(1) &= -6 \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^{n-1}(n - 1)!\,x^{-n} & f^{(n)}(1) &= (-1)^{n-1}(n - 1)! \\
&\vdots & &\vdots
\end{aligned}$$

Hence,

$$f(x) = f(1) + \frac{f'(1)}{1!}(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 + \frac{f^{(4)}(1)}{4!}(x - 1)^4 + \cdots$$

$$= 0 + (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{2}{6}(x - 1)^3 - \frac{6}{24}(x - 1)^4 + \cdots$$

$$= (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 + \frac{1}{4}(x - 1)^4 + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x - 1)^n}{n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x - 1)^{n+1}}{n + 1} \quad \text{for } 0 < x \le 2$$

Verify the domain by using the Ratio Test.

5. Find the Maclaurin Series for $f(x) = \dfrac{x^3}{4 + 9x^2}$.

Method 1: (Geometric Series)

$$f(x) = \frac{x^3}{4 + 9x^2} = \frac{x^3}{4} \cdot \frac{1}{1 - \left(-\frac{9}{4}x^2\right)}$$

$$= \frac{x^3}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{9}{4}x^2\right)^n$$

So,

$$f(x) = \frac{x^3}{4} \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n}}{2^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n+3}}{2^{2n+2}}$$

Since this series came from a geometric series the domain (or interval of convergence) is:

$$\left|\frac{9}{4}x^2\right| < 1$$

or

$$|x^2| < \frac{4}{9}$$

or

$$|x| < \frac{2}{3}$$

or

$$-\frac{2}{3} < x < \frac{2}{3}$$

Method 2: (Taylor's Theorem)

Let $g(x) = \dfrac{1}{1 + x}$. Then $f(x) = \frac{x^3}{4} g\left(\frac{9x^2}{4}\right)$.

$$g(x) = (1 + x)^{-1} \qquad g(0) = 1$$

$$g'(x) = -(1 + x)^{-2} \qquad g'(0) = -1$$

$$g''(x) = 2(1 + x)^{-3} \qquad g''(0) = 2$$

$$g'''(x) = -6(1 + x)^{-4} \qquad g'''(0) = -6$$

$$\vdots \qquad \vdots$$

$$g^{(n)}(x) = (-1)^n n!\,(1 + x)^{-n-1} \qquad g^{(n)}(0) = (-1)^n n!$$

So,

$$g(x) = g(0) + \frac{g'(0)}{1!}(x - 0) + \frac{g''(0)}{2!}(x - 0)^2 + \frac{g'''(0)}{3!}(x - 0)^3 + \frac{g^{(4)}(0)}{4!}(x - 0)^4 + \cdots$$

$$= 1 - x + \frac{2}{2}x^2 + \frac{-6}{6}x^3 + \frac{24}{24}x^4 + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

Thus,

$$f(x) = \frac{x^3}{4} g\left(\frac{9x^2}{4}\right)$$

$$= \frac{x^3}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{9x^2}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n+3}}{2^{2n+2}} \quad \text{for } -\frac{2}{3} < x < \frac{2}{3}$$

Verify the domain by the Ratio Test!

6. Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

Using Taylor's Theorem, we have

$$\begin{aligned}
f(x) &= (1 + x)^k \\
f'(x) &= k(1 + x)^{k-1} \\
f''(x) &= k(k - 1)(1 + x)^{k-2} \\
f'''(x) &= k(k - 1)(k - 2)(1 + x)^{k-3} \\
&\vdots \\
f^{(n)}(x) &= k(k - 1)\cdots(k - n + 1)(1 + x)^{k-n}
\end{aligned}$$

For $x = 0$, we find that

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= k \\
f''(0) &= k(k - 1) \\
f'''(0) &= k(k - 1)(k - 2) \\
&\vdots \\
f^{(n)}(0) &= k(k - 1)\cdots(k - n + 1)
\end{aligned}$$

Therefore, the Maclaurin series for $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

This series is called the **binomial series**. Using the Ratio Test, this series will converge if $|x| < 1$ and diverge is $|x| > 1$. (Verify this!) Convergence at the endpoints, ± 1, depends on the value for k.

The traditional notation for the coefficients of the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the **binomial coefficients**.

THE BINOMIAL SERIES: If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$$

The series converges at the endpoint 1 if $-1 < k \le 0$. The series converges at both endpoints, ± 1, if $k \ge 0$.

7. Represent $\sqrt{1+x}$ as a Maclaurin series.

Using the Binomial Series above, we find that

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!} x^4 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^5 + \cdots$$

Please note the table of important Maclaurin series on page 779 of the text.

Exercises:

1. Find the Maclaurin series for $f(x) = x\cos(3x)$. Use the power series for $\cos x$ we developed in Example 2 on page 2. State the domain.

2. Find the Maclaurin series for $g(x) = \arctan x$. Use this result to find the Maclaurin series for $f(x) = \dfrac{\arctan(x^3)}{x}$. State the domain.

3. Find the Taylor series for $f(x) = \ln(2x + 3)$ about $a = -1$. State the domain.

4. Find the Maclaurin series for $f(x) = x^2\sin(x^2)$. Use the power series for $\sin x$ we developed in Example 1 on page 2. State the domain.

Answers:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n+1}}{(2n)!}$, all x

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $-1 \le x \le 1$; $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{2n+1}$, $-1 \le x \le 1$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} (x+1)^{n+1}}{n+1}$, $-\frac{3}{2} \le x \le \frac{3}{2}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)!}$, all x