Section 12.1: Sequences

Introduction:

A ***sequence*** $\{a_n\}$ is a function whose domain is the set of positive integers.
The function values $a_1, a_2, a_3, \ldots, a_n, \ldots$ are called the ***terms*** of the sequence.
The ***value*** of the function at the integer n is a_n.
The variable n is called the ***index***.

A sequence may be specified in three ways:

- By an explicit formula,
  $a_n = 2(3^n)$
- By a recursive formula, and
  $$a_{n+1} = 3a_n, \quad a_1 = 6$$
- By giving enough terms to establish a pattern.
  6, 18, 54, 162, . . .

Graphing a Sequence on the TI-83/84:

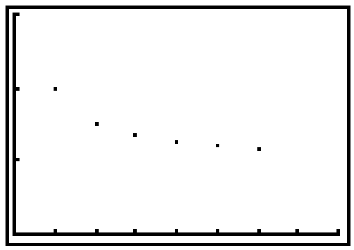
1. Press **MODE.** Select **Seq** at the end of the fourth line.
2. Press **Y=** and type in the sequence. Use the ***X,T,θ,n*** key to get the variable n.
3. Adjust the viewing window as necessary.

Example: Graph the sequence $b_n = 1 + \dfrac{1}{n}$.

The Limit of a Sequence:

Consider the sequence $b_n = 1 + \frac{1}{n}$.

$b_1 = 2, b_2 = 3/2, b_3 = 4/3, b_4 = 5/4, b_5 = 6/5, b_6 = 7/6.$



It appears that $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right) = 1$.

NOTE: All limit theorems for functions (learned in Calculus I) also apply to sequences.

Definition: If $\lim_{n \to \infty} a_n = L$ is finite, then the sequence $\{a_n\}$ ***converges***; if $\lim_{n \to \infty} a_n = L$ is infinite or does not exist, then the sequence $\{a_n\}$ ***diverges***.

Observe that if p is a positive number $\lim_{n \to \infty} \frac{1}{n^p} = 0$.

Examples: Show that the following sequences converge.

1. $a_n = \dfrac{n^2 + 3n}{2n^2 + 1}$

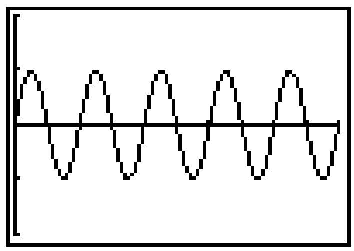
2. $b_n = \dfrac{3n - 1}{n + \sqrt{n}}$

Theorem: Let $\{a_n\}$ be a sequence and let f be a function defined on $[1,\infty)$ such that $f(n) = a_n$ for $n = 1, 2, 3, \ldots$.

$$\text{If } \lim_{x \to \infty} f(x) = L, \text{ then } \lim_{n \to \infty} a_n = L.$$

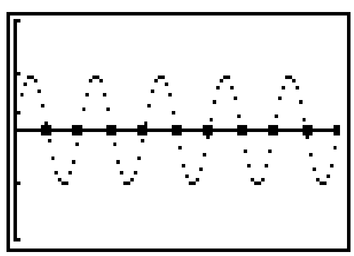
However, it is not necessarily true that if $\lim_{n \to \infty} a_n = L$, then $\lim_{x \to \infty} f(x) = L$. For example:

If $f(x) = \sin \pi x$, then $\lim_{x \to \infty} \sin \pi x$ does not exists. See the graph below.



The graph of
$y = \sin \pi x$.

If $a_n = \sin \pi n$, then $\lim_{n \to \infty} \sin \pi n = 0$. This is because $\sin(n\pi)$ is a succession of zeros for $n = 1, 2, 3, \ldots$. See the graph below.



The graph of
$a_n = \sin \pi n$
with $y = \sin \pi x$
superimposed.

Example: Does $\left\{\frac{\ln n}{e^n}\right\}$ converge?

<u>Some Convergence Theorems</u>:

The Squeeze Theorem: Suppose that $\{a_n\}$ and $\{c_n\}$ both converge to L and that $a_n \le b_n \le c_n$ for $n \ge K$ (K is a <u>fixed</u> positive integer). Then $\{b_n\}$ converges to L.

Example: Use the Squeeze Theorem to show that $\left\{\frac{\sin^3 n}{n}\right\}$ converges.

Absolute Value Theorem: If $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$.

Example: Show that if $-1 < r < 1$, then the sequence $\{r^n\}$ converges.

Solution:

If $r = 0$, $\lim_{n \to \infty} 0^n = 0$. So, we only need to deal with the case when $-1 < r < 0$ and $0 < r < 1$; that is, when $0 < |r| < 1$.

Since $|r| < 1$, $\frac{1}{|r|} > 1$. Thus, there is some positive number p such that $\frac{1}{|r|} = 1 + p$.

Now, we recall the Binomial Formula which says

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-2}b^3 + \cdots + nab^{n-1} + b^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k.$$

Note that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Using the Binomial Formula, we see that

$$\frac{1}{|r^n|} = \frac{1}{|r|^n} = (1 + p)^n = 1 + np + \text{other positive terms}$$

$$\geq np.$$

Hence, we know that

$$0 \leq |r^n| \leq \frac{1}{np}$$

and that

$$\lim_{n \to \infty} 0 = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{np} = 0.$$

So, by the Squeeze Theorem, $\lim_{n \to \infty} |r^n| = 0$. And, by the theorem above $\lim_{n \to \infty} r^n = 0$.

Monotonic Sequences:

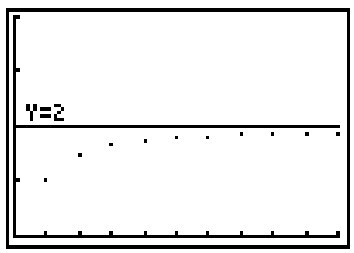
A sequence $\{a_n\}$ is called ***nondecreasing*** if $a_n \le a_{n+1}$, $n \ge 1$.
A sequence $\{b_n\}$ is called ***nonincreasing*** if $b_n \ge b_{n+1}$, $n \ge 1$.
A sequence is ***monotonic*** if it is either nondecreasing or nonincreasing.

Two examples of nondecreasing sequences are

$$a_n = n^2 \qquad \text{and} \qquad b_n = 1 - \frac{1}{n}.$$

A nondecreasing sequence can do one of two things:

1. March off to infinity, or
2. If it is bounded above (that is, $a_n \le K$ for $n \ge 1$ and some fixed number K), then it must bump against a "lid." See the diagram below.



A nondecreasing sequence that is bounded above by 2.

NOTE: Sequence $\{a_n\}$ above marches off to infinity. However, sequence $\{b_n\}$ above is bounded above by 1 and has limit 1.

Bounded Sequences:

1. A sequence $\{a_n\}$ is ***bounded above*** if there is a real number M such that $a_n \le M$ for all n. The number M is called an ***upper bound*** of the sequence.
2. A sequence $\{a_n\}$ is ***bounded below*** if there is a real number N such that $a_n \ge N$ for all n. The number N is called an ***lower bound*** of the sequence.
3. A sequence $\{a_n\}$ is ***bounded*** if it is bounded above and bounded below.

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

NOTE: In the theorem above it is not necessary that the sequence $\{a_n\}$ be monotonic initially, only that they be monotonic from some point on—that is, for $n \geq K$. In fact, *the convergence or divergence of a sequence does not depend on the character of its initial terms but rather on what is true for large n.*

Example: Use the Monotonic Sequence Theorem to show that

$$a_n = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right), \quad n \geq 2$$

converges.