Function Operations and Inverses

The definitions of adding, subtracting, multiplying, and dividing functions are natural and intuitive. Let f and g be functions.

$(f+g)(x)=f(x)+g(x)$
$(f-g)(x)=f(x)-g(x)$
$(f\cdot g)(x)=f(x)\cdot g(x)$
$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)},\quad g(x)\neq 0$

Simply add, subtract, multiply, or divide the rules given for each function, paying attention to the order on both subtraction and division.

(1) For example, if f(x) = x + 1 and g(x) = 2x – 1,
(f + g)(x) = (x + 1) + (2x – 1) = 3x, and
(f + g)(5) = (5 + 1) + (2(5) – 1) = 6 + 9 = 15

[Of course, the general rule would be an easier method: (f + g)(5) = 3(5) = 15.]

The other 3 operations are just as straightforward.

(f – g)(x) = (x + 1) – (2x – 1) = (x + 1) + (-2x + 1) = -x + 2, and
(f – g)(5) = (5 + 1) – (2(5) – 1) = 6 – 9 = -3 or
(f – g)(5) = -5 + 2 = -3

$$(f\cdot g)(x)=(x+1)(2x-1)=2x^2-x+2x-1=2x^2+x-1$$

$\left(\frac{f}{g}\right)(x)=\frac{x+1}{2x-1}$ The domain of $\frac{f}{g}$ is $x\neq\frac{1}{2}$ since 2x – 1 cannot be 0. (Division by zero is undefined.)

Now we will introduce another operation called **composition**. Let's consider the three measurement units for temperature. The Celsius temperature depends on the Fahrenheit temperature according to the function, $C=\frac{9}{5}(F-32)$. The Kelvin temperature depends on the Celsius temperature according to the function, K = C + 273.15. We could, in a sense, substitute the right-hand side of the Celsius-Fahrenheit formula for C in the Kelvin-Celsius formula to obtain $K=\frac{9}{5}(F-32)+273.15$ or, after distributing and combining like terms, $K=\frac{9}{5}F+215.55$. Now we have Kelvin temperature dependent on Fahrenheit temperature, or Kelvin as a function of Fahrenheit. This process is called composition.

More formerly, the composition of two functions f and g is defined as

$(f\circ g)(x)=f(g(x))$	f composed with g or f composed on g

The following examples should help.

(2) For the two functions given above [f(x) = x + 1 and g(x) = 2x – 1],

$$(f \circ g)(x) = f(g(x)) = f(2x - 1) = (2x - 1) + 1 = 2x$$

We're basically substituting the formula for g(x), and then applying the rule for f on the result.

(3) Let $f(x) = x^2$ and $g(x) = x - 3$. Find $(f \circ g)(0)$, $(f \circ g)(3)$, $(f \circ g)(-1)$, and $(f \circ g)(x)$.

$$(f \circ g)(0) = f(g(0)) = f(0 - 3) = f(-3) = (-3)^2 = 9,$$

$$(f \circ g)(3) = f(g(3)) = f(3 - 3) = f(0) = (0)^2 = 0,$$

$(f \circ g)(-1) = f(g(-1)) = f(-1 - 3) = f(-4) = (-4)^2 = 16$, and, in general,

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2 = (x - 3)(x - 3) = x^2 - 6x + 9.$$

As with subtraction and division of functions, the order of composition is very important. The next example is chosen to illustrate this fact.

(4) Let $f(x) = x^2$ and $g(x) = x - 3$. Find $(g \circ f)(0)$, $(g \circ f)(3)$, $(g \circ f)(-1)$, and $(g \circ f)(x)$.

$$(g \circ f)(0) = g(f(0)) = g(0^2) = g(0) = 0 - 3 = -3,$$

$$(g \circ f)(3) = g(f(3)) = g(3^2) = g(9) = 9 - 3 = 6,$$

$(g \circ f)(-1) = g(f(-1)) = g((-1)^2) = g(1) = 1 - 3 = -2$, and, in general,

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3.$$

Notice that, for these two functions f and g, $(g \circ f)(x) \neq (f \circ g)(x)$ for any x value we tried, and they certainly are not generally equivalent. There is a value for which $(g \circ f)(x) = (f \circ g)(x)$. Check out x = 2. You will find that $(g \circ f)(2) = (f \circ g)(2)$.

Let's see a few more examples with a wider variety of function types.

(5) Let $f(x) = |x|$ and $g(x) = \sqrt{x + 2}$. Find $(f \circ g)(-3)$, $(f \circ g)(2)$, $(g \circ f)(-3)$, and $(g \circ f)(x)$.

$(f \circ g)(-3) = f(g(-3)) = g(\sqrt{-3 + 2}) = g(\sqrt{-1})$. Since $\sqrt{-1}$ isn't real, we cannot proceed further; there is no solution because of the domain issue.

$$(f \circ g)(2) = f(g(2)) = f(\sqrt{2 + 2}) = f(\sqrt{4}) = f(2) = |2| = 2,$$

$(g \circ f)(-1) = g(f(-1)) = g\left((-1)^2\right) = g(1) = 1 - 3 = -2$, and, in general,

$(g \circ f)(x) = g(f(x)) = g\left(x^2\right) = x^2 - 3.$

This example also illustrates a domain issue. The domain of $(g \circ f)(x)$ is clearly all real numbers. The composition of f on g, however, is $(f \circ g)(x) = \left|\sqrt{x + 2}\right|$, and the domain of this function involves x + 2 ≥ 0, leading to x ≥ -2. This explains why $(f \circ g)(-3)$ cannot be done; the domain of $f \circ g$ doesn't include -3.

(6) Let $c(x) = x^3$ and $r(x) = \dfrac{1}{x + 3}$. Find $(c \circ r)(x)$, $(r \circ c)(x)$, and the domain of each function.

$(c \circ r)(x) = c(r(x)) = c\left(\dfrac{1}{x + 3}\right) = \left(\dfrac{1}{x + 3}\right)^3 = \dfrac{1}{(x + 3)^3}$. The domain comes from x + 3 ≠ 0, which yields x ≠ -3.

$(r \circ c)(x) = r(c(x)) = r\left(x^3\right) = \dfrac{1}{x^3 + 3}$. The domain comes from $x^3 + 3 \neq 0$, which leads to $x^3 \neq -3$ or $x \neq \sqrt[3]{-3}$. Here again, the reversal of the composition order gives different results.

We'd like now to shift direction a bit. The next example shows the process of **decomposition**, a very important skill for calculus coursework. We are "breaking down" a function into parts that are easier to handle.

(7) If $f(x) = x^3$, $g(x) = x + 2$, and $h(x) = \sqrt{x}$, write each given function as a composition using two of the given functions.

(a) $F(x) = x^3 + 2$ $\quad$ $(g \circ f)(x) = g(f(x)) = g\left(x^3\right) = x^3 + 2$

(b) $G(x) = \sqrt{x + 2}$ $\quad$ $(h \circ g)(x) = h(g(x)) = h(x + 2) = \sqrt{x + 2}$

(c) $H(x) = \sqrt{x^3}$ $\quad$ $(h \circ f)(x) = h(f(x)) = h\left(x^3\right) = \sqrt{x^3}$

We've studied functions throughout this algebra course. A function is a special relation in which every element of the domain of x values is paired with exactly one element in the range of y values. When the reverse is also true, we have a special type of function, a one-to-one function.

A **one-to-one function** is a relation in which every x value corresponds with only one y value and every y value corresponds with only one x value.

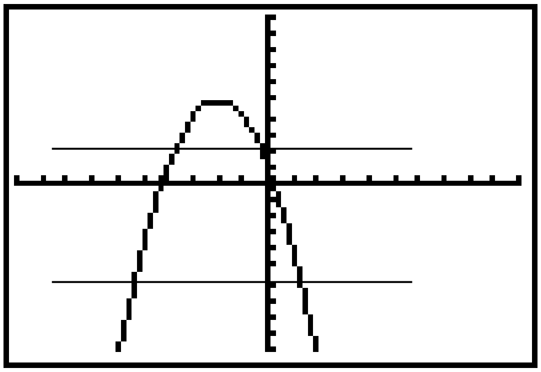
Examples: f = {(6, 2), (5, 4), (-1, 0), (7, 3)} $\quad$ g = {(1, 1), (5, 5), (10, 10), (-5, -5)}

One of the natural results of the one-to-one definition is a visual test to determine "one-to-one-ness".

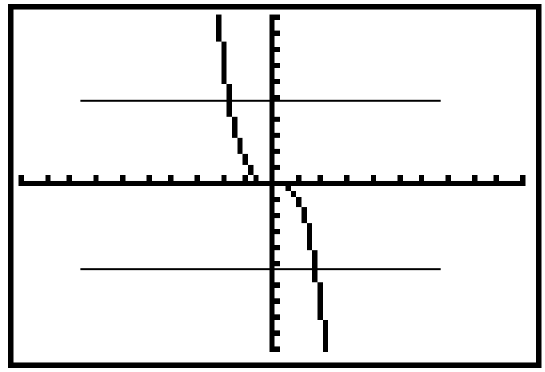
You may recall that the vertical line test is used to determine whether a relation is a function; a similar test may be used to determine whether a function is one-to-one.

Horizontal Line Test:
If every horizontal line intersects the graph of a function at most once, then the function is one-to-one.

Examples:



A horizontal lines crosses through two points on the graph, so the function is <u>not</u> one-to-one.



No horizontal line crosses through two points on the graph, so the function is one-to-one.

One of the important features of one-to-one functions is that all one-to-one functions have inverses that are functions, as well.

Finding the inverse of a function is as simple as switching coordinates. One feature of **inverse functions** is if the point (a, b) is contained in a function f, the point (b, a) must be contained in its inverse, denoted by the symbol, f^{-1}, and read "f inverse". For example, if we are given the function f = {(6, 2), (5, 4), (-1, 0), (7, 3)}, its inverse is f^{-1} = {(2, 6), (4, 5), (0, -1), (3, 7)}.

If we're given a function rule such as f(x) = x + 3, there is a 4-step process to find its inverse. An example is provided to make the steps clearer.

1. Replace f(x) with y.	$y = x + 3$
2. Interchange x and y.	$x = y + 3$
3. Solve for y.	$x - 3 = y$ or $y = x - 3$
4. Replace y with f^{-1}.	$f^{-1}(x) = x - 3$

(8) Find the inverse of g(x) = 2x – 4.

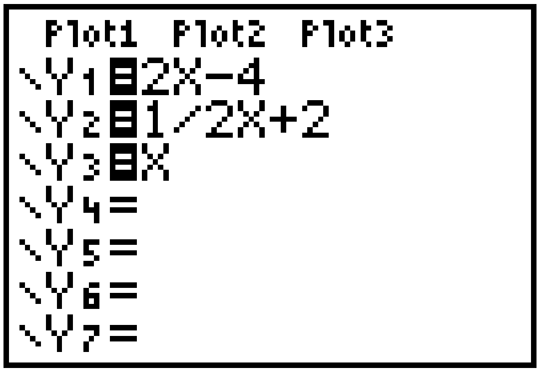
$y = 2x - 4$

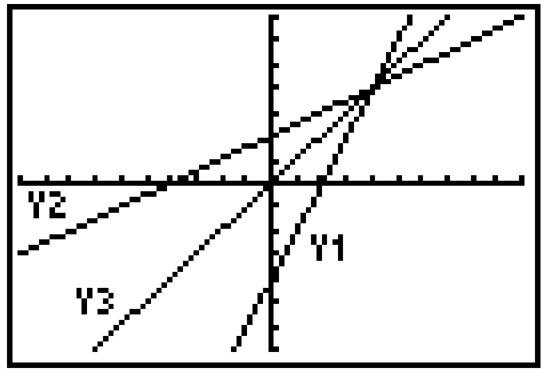
$x = 2y - 4$
$x + 4 = 2y$
$y = \frac{x + 4}{2}$ or $y = \frac{1}{2}x + 2$
$g^{-1}(x) = \frac{1}{2}x + 2$

(9) Find the inverse of $h(x) = \frac{1}{x + 3}$

$y = \frac{1}{x + 3}$, then $x = \frac{1}{y + 3}$
Using cross products, we have $x(y + 3) = 1$. Dividing both sides by x, we have $y + 3 = \frac{1}{x}$; then subtracting 3 on both sides, we have $y = h^{-1}(x) = \frac{1}{x} - 3$.

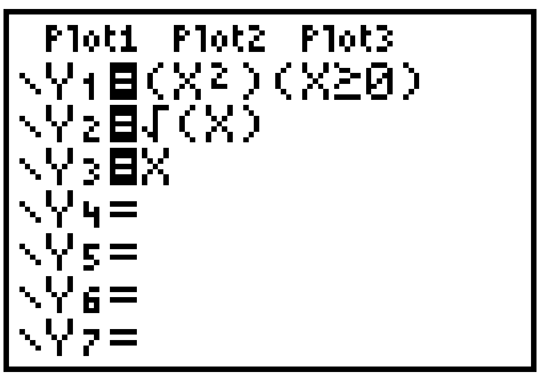
For example (8), we will graph both functions using the window [-10, 10, 1, -7, 7, 1], along with the identity function, y = x.

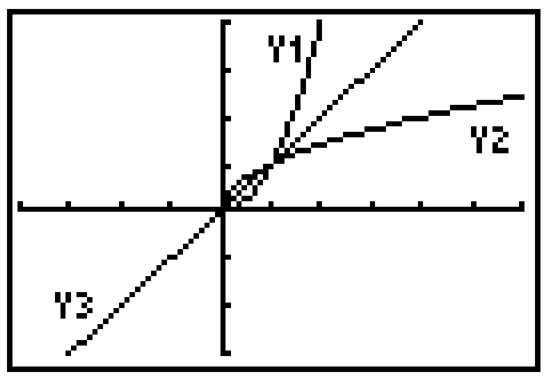




One thing to notice is that if the window is well chosen (or you use a piece of graph paper), the graphs of the functions f and f^{-1} are mirror reflections of one another over the line y = x. This is generally true for all inverse functions.

For example, the graph of $f(x) = x^2$ with $x \ge 0$ has an inverse, $f^{-1}(x) = \sqrt{x}$ (with domain also $x \ge 0$). Here, we restrict the domain of f so that f is a one-to-one function, because then its inverse is a function, as well. Many of these inverses are paired on typical calculator keys. The graphs of $f(x) = x^2$ with $x \ge 0$ and $f^{-1}(x) = \sqrt{x}$ with $x \ge 0$ are shown below.





One other significant fact about inverses involves composition. (We've come full circle!) To determine whether functions are indeed inverses, we may check whether the following compositions hold:

$$f\left(f^{-1}(x)\right) = x \quad \underline{\text{and}} \quad f^{-1}\left(f(x)\right) = x$$

In other words, if you perform a function on a given value, and then perform the inverse function on the resulting value, you should end up with the value you started with in the first place. Again, a few examples should clear things up nicely.

(10) Verify algebraically that g(x) = 2x – 4 and $g^{-1}(x) = \frac{1}{2}x + 2$ are inverses.

$$g\left(g^{-1}(x)\right) = g\left(\frac{1}{2}x + 2\right) = 2\left(\frac{1}{2}x + 2\right) - 4 = (x + 4) - 4 = x$$

$$g^{-1}(g(x)) = g^{-1}(2x - 4) = \frac{1}{2}(2x - 4) + 2 = (x - 2) + 2 = x$$

Since both compositions result in x, the two functions must be inverses of one another.

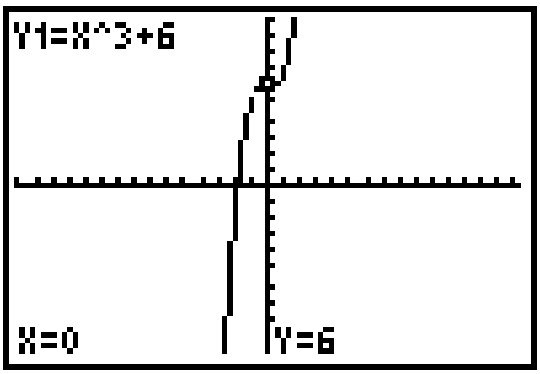
(11) Verify algebraically that $f(x) = x^3 + 6$ and $f^{-1}(x) = \sqrt[3]{x - 6}$ are inverses.

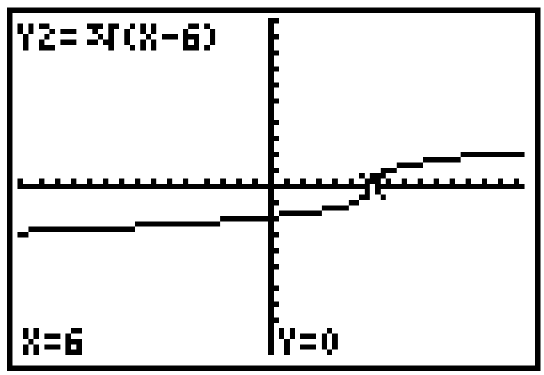
$$f(f^{-1}(x)) = f(\sqrt[3]{x - 6}) = (\sqrt[3]{x - 6})^3 + 6 = (x - 6) + 6 = x$$

$$f^{-1}(f(x)) = f^{-1}(x^3 + 6) = \sqrt[3]{(x^3 + 6) - 6} = \sqrt[3]{x^3} = x$$

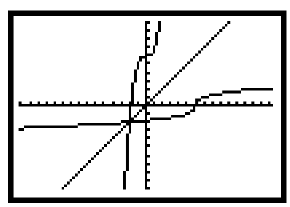
Both compositions result in the original value, x, so they are inverses of each other.

We include the two graphs with the TI-83 square window:





Note: Y1 above is f(x), Y2 is $f^{-1}(x)$, and they are shown together with y = x below: Notice the mirror reflection relationship between the graphs.



Exercises:

1. If $f(x) = x^2 + 2$ and $g(x) = 3x - 2$, find the following values.

(a) $(f + g)(1)$ (b) $(f - g)(-1)$ (c) $(f \cdot g)(0)$

(d) $(g \circ f)(-1)$ (e) $\left(\frac{f}{g}\right)(4)$

2. If $f(x) = \frac{x + 2}{x - 3}$ and $g(x) = \sqrt{x + 4}$, find the following values.

(a) $(f + g)(0)$ (b) $(f - g)(4)$ (c) $(f \cdot g)(0)$

(d) $(g \circ f)(4)$ (e) $\left(\frac{f}{g}\right)(-3)$

3. If $f(x) = \dfrac{x + 2}{x - 3}$ and $g(x) = \sqrt{x + 4}$, find (if possible)

(a) $(f \circ g)(0)$ (b) $(f \circ g)(-3)$ (c) $(f \circ g)(5)$
(d) $(g \circ f)(2)$ (e) $(g \circ f)(7)$

4. If $f(x) = x^2 + 2$ and $g(x) = 3x - 2$, find

(a) $(f \circ g)(0)$ (b) $(f \circ g)(-3)$ (c) $(f \circ g)(5)$
(d) $(g \circ f)(2)$ (e) $(g \circ f)(7)$

5. If $f(x) = x^2 + 2$ and $g(x) = 3x - 2$, find

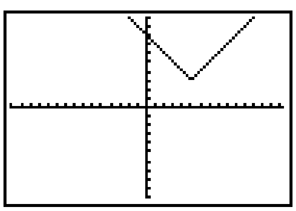
(a) $(f \circ g)(x)$ and its domain
(b) $(g \circ f)(x)$ and its domain

6. If $f(x) = \dfrac{x + 2}{x - 3}$ and $g(x) = \sqrt{x + 4}$, find

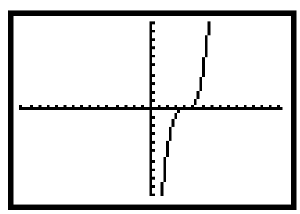
(a) $(f \circ g)(x)$ and its domain
(b) $(g \circ f)(x)$ and its domain

7. Which of the following represent one-to one functions? You may want to enter the function in your graphing calculator to help you decide.

(a) {(0, 2), (1, 3), (3, 4), (4, 3)}
(b) $y = (x \square 2)^3$ (c) $y = \sqrt{|x|}$
(d)

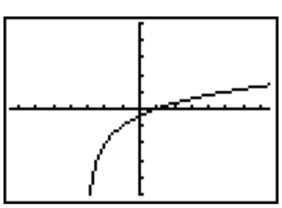


(e)

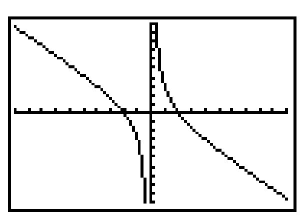


8. Which of the following represent one-to one functions? You may want to enter the function in your graphing calculator to help you decide.

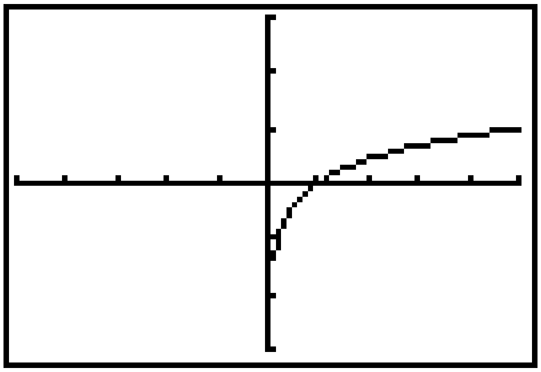
(a) {(0, 1), (1, 3), (3, 4), (4, 7)}
(b) $y = (x \square 2)^2$ (c) $y = x^{2/3} + 4$
(d)



(e)



9. The following graph is a one-to-one function. Label it f. Draw the line y = x and the inverse of this function. Label it f^{-1}.



10. Find the inverse of each given function.

(a) {(0, 1), (1, 3), (3, 4), (4, 7)}

(b) $y = 2x + 3$

(c) $y = (x - 2)^3$

(d) $y = 2^x$

(e) $y = x^2 - 3, \ x \geq 0$

(f) $y = \frac{4}{x}$

11. Find the inverse of each given function.

(a) {(0, 5), (1, -3), (3, -2), (4, 1), (2, 2)}

(b) $y = -3x + 6$

(c) $y = (x + 1)^3$

(d) $y = 3^x$

(e) $y = (x - 3)^2, \ x \geq 0$

(f) $y = \frac{4}{x^2}, \ x > 0$

12. Consider $f(x) = 2x - 6$.

(a) Make a table of values (at least 3 points) for this function and its inverse.
(b) Find a formula for the inverse of this function.
(c) Graph f(x) and its inverse on the same set of axes, along with the graph of y = x.

13. Consider $f(x) = x^3 + 2$.

(a) Make a table of values (at least 3 points) for this function and its inverse.
(b) Find a formula for the inverse of this function.
(d) Graph f(x) and its inverse on the same set of axes, along with the graph of y = x.

14. The inverse of $f(x) = \frac{2}{x + 1}$ is $g(x) = \frac{2 - x}{x}$. Show that g "undoes" the effect of f on x = 3.

15. Find each of the following.

(a) $(f \circ f^{-1})(3.5)$

(b) $(f^{-1} \circ f)(\sqrt{2})$

(c) $f(f^{-1}(-5))$

(d) $f^{-1}(f(L))$

Solutions:

1. (a) 4 (b) 8 (c) –4 (d) –8 (e) 1.8

2. (a) $1\frac{1}{3}$ (b) $6 - 2\sqrt{2}$ (c) $-1\frac{1}{3}$ (d) $2\sqrt{2} - 6$ (e) $\frac{1}{6}$

3. (a) –4 (b) –1.5 (c) no solution (d) 0 (e) 2.5

4. (a) 6 (b) 123 (c) 171 (d) 16 (e) 151

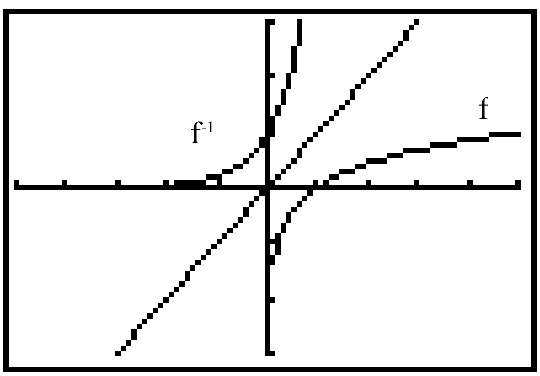
5. (a) $(f \circ g)(x) = 9x^2 - 12x + 6$, all reals
   (b) $(g \circ f)(x) = 3x^2 + 4$, all reals

6. (a) $(f \circ g)(x) = \dfrac{\sqrt{x+4} + 2}{\sqrt{x+4} - 3}$, $x \ge -4, x \ne 5$
   (b) $(g \circ f)(x) = \sqrt{\dfrac{x+2}{x-3} + 4}$, $x \,\square\, 2$ or $x > 3$

7. (a) No (b) Yes (c) No (d) No (e) Yes

8. (a) Yes (b) No (c) No (d) Yes (e) No

9.



y = x

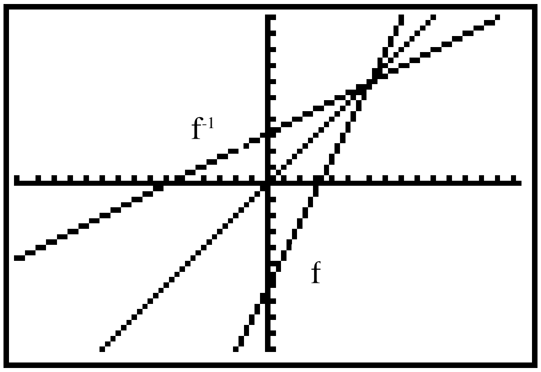
10. (a) {(1, 0), ((3, 1), ((4, 3), (7, 4)} (b) $y = \dfrac{x - 3}{2}$ or $y = \frac{1}{2}x - \frac{3}{2}$
    (c) $y = \sqrt[3]{x} + 2$ (d) $x = 2^y$ (or $y = \log_2 x$)
    (e) $y = \sqrt{x+3}$ (f) $y = \dfrac{4}{x}$

11. (a) {(5, 0), ((-3, 1), ((-2, 3), (1, 4), (2, 2)} (b) $y = \dfrac{6 - x}{3}$ or $y = -\frac{1}{3}x + 2$
    (c) $y = \sqrt[3]{x} - 1$ (d) $x = 3^y$ (or $y = \log_3 x$)
    (e) $y = \sqrt{x} + 3$ (f) $y = \dfrac{2}{\sqrt{x}}$

12. (a) Answers may vary.

(b) $f^{-1}(x) = \frac{x + 6}{2}$ or $y = \frac{1}{2}x + 3$

(c)

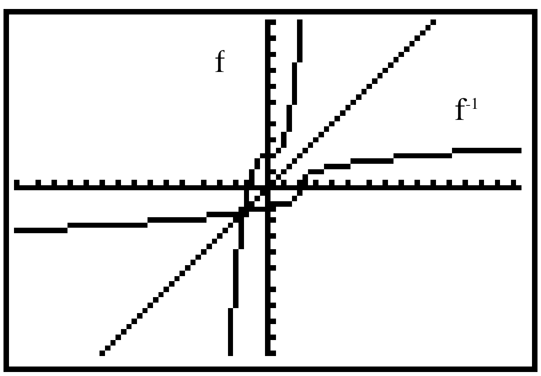


y = x

13. (a) Answers may vary.

(b) $f^{-1}(x) = \sqrt[3]{x - 2}$

(c)



y = x

14. $f(3) = \frac{1}{2}$ and $g\left(\frac{1}{2}\right) = 3$

15. (a) 3.5 (b) $\sqrt{2}$ (c) –5 (d) L

Do your best! Live and learn!