Logarithmic Functions

Another common type of non-linear function is the **logarithmic function**. By definition, the logarithmic function is directly related to the exponential function; the two functions are called i**nverses** of one another, much like $y = \pm\sqrt{x}$ is the inverse of $y = x^2$.

An equation for the inverse of an exponential function can be obtained by "switching the places of x and y" as shown below:

Exponential Function: $y = b^x$ (Recall that $b > 0$, $b \neq 1$)

Inverse function: $x = b^y$ (This is called **exponential form** for a logarithmic function.)

Taking this inverse function and "solving for y", we create a new "powerful" term. The equation tells us that *y is the exponent on b that results in x*. The words "logarithm" and "exponent" are interchangeable, so we may say that *y is the logarithm on b that results in x*. This description can be abbreviated as

$y = \log_b x$ (This is called **logarithmic form** for a logarithmic function.)

which reads "y equals log, base b, of x" or "y equals log of x, base b".

For any constant $b > 0$, $b \neq 1$, the equation $y = \log_b x$ defines a logarithmic function with base b and domain all $x > 0$.

Think of exponential form ($x = b^y$) and logarithmic form ($y = \log_b x$) as two sides of the very same coin. Converting a logarithmic function from logarithmic form to exponential form (and vice versa) is helpful for graphing, as we will see in this first example.

Examples:

(1) Graph $y = \log_2 x$.

Solution: Rewrite the equation in exponential form, $x = 2^y$, and then input values for y and find the corresponding x value.

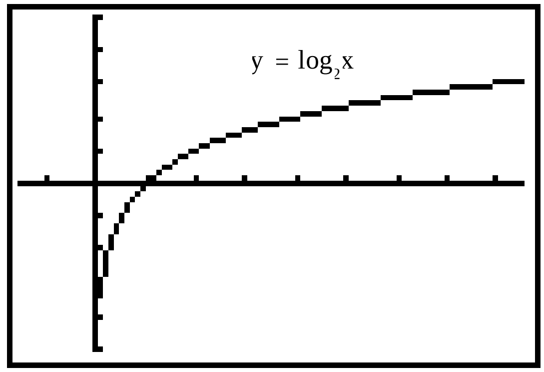
For example, when $y = 3$, $x = 2^3 = 8$,

and when $y = -2$, $x = 2^{-2} = \frac{1}{4}$ or 0.25.

x	y
$^1/_{16}$ or 0.0625	-4
$^1/_8$ or 0.125	-3
$^1/_4$ or 0.25	-2
$^1/_2$ or 0.5	-1
1	0
2	1
4	2
8	3
16	4

You may notice that while the graph of $y = 2^x$ includes points such as (-1, 0.5), (0, 1) and (1, 2), this logarithmic function contains (0.5, -1), (1, 0), and (2, 1); this goes back to the idea that if the point represented by the ordered pair (a, b) is on the graph of a function, (b, a) will be on the graph of its inverse. Also, TI calculators have a "DrawInv" command that enables you to draw the inverse of any given function. We will learn log rules that will make entering a function like $y = \log_2 x$ even more straightforward.

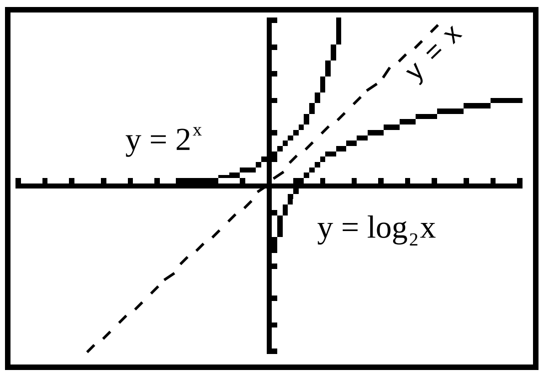
Next, we plot the points in our table, and then draw a continuous curve that contains the points.



Reviewing the relationship between the two forms,

$\log_2 16 = 4$ because $2^4 = 16$;
$\log_2 8 = 3$ because $2^3 = 8$;
$\log_2 2 = 1$ because $2^1 = 2$;
$\log_2 1 = 0$ because $2^0 = 1$; and
$\log_2\left(\frac{1}{2}\right) = -1$ because $2^{-1} = \frac{1}{2}$.

Since the logarithmic function and the corresponding exponential function are inverses, the two graphs are directly related; one is the mirror reflection of the other over the graph of the identity function, y = x.



The chart below shows the critical functions we consider in Sections 5.2, 5.3, 5.4, and 5.5.

Logarithmic Functions		**Exponential Functions**
Logarithmic form	**Exponential form**	
$y = \log_b x$	$b^y = x$	$y = b^x$

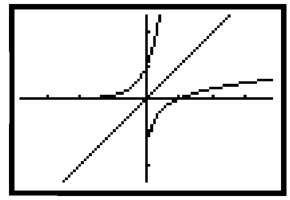
↑ These are equivalent forms. ↑

Using methods similar to Example 1, we can graph the logarithmic function $y = \log_{10} x$, along with its inverse, $y = 10^x$. The function $y = \log_{10} x$, often abbreviated as y = log x, is called the **common logarithm** because of the common use of base 10 in numeration and in metric measurement.

[You'll see the LOG key on most scientific or graphing calculators, and you can directly enter y = log x into your graphing calculator in order to graph this logarithmic function.]

And here are the exponential and logarithmic functions, base 10. Points on the graph of y = log x include (0.01, –2), (0.1, –1), (1, 0), (10, 1), and (100, 2).

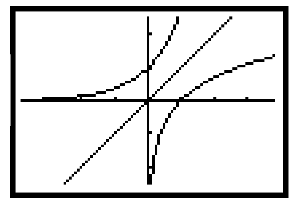
x	y
0.01	-2
0.1	-1
1	0
10	1
100	2



We can also graph the logarithmic function $y = \log_e x$ together with its inverse, $y = e^x$. The function $y = \log_e x$, often abbreviated as y = ln x, is called the **natural logarithm** because of the use of the base e in several natural applications. [You'll see the LN key on most scientific or graphing calculators, and you can directly enter y = ln x into your graphing calculator in order to graph this logarithmic function.]

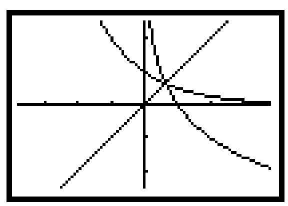
Points on the graph of y = ln x include (0.01, –4.605), (0.1, –2.303), (1, 0), (2, 0.693), (10, 2.303), and (100, 4.605). Several of these y-coordinates are approximations of irrational numbers.

x	y
0.01	-4.605
0.1	-2.303
1	0
2	0.693
3	1.099
4	1.386
10	2.303
100	4.605



The three examples we've considered are of the form $y = \log_b x$ with bases 2, 10, and e. Notice that whenever b is greater than 1, the graph will increase from left to right.

For the sake of completeness, we include the graph of $y = \log_{1/2} x$ along with its inverse, $y = \left(\frac{1}{2}\right)^x$ and the identity function, y = x.



Here is a summary of some of the general features of the graphs of logarithmic functions:

- The curves are continuous, increasing over their entire domains if b > 1 and decreasing if 0 < b < 1.
- The domain is $\{x \mid x > 0\}$, and the range includes all real numbers. These facts are important for basic calculations and for solving equations.
- Each logarithmic function has no y-intercept, and their x-intercepts are (1, 0) since $\log_b 1 = 0$ for any base b.
- The vertical line x = 0 (y-axis) is the only asymptote for the curve.

One of the real keys in dealing with logarithms is freely going back and forth between logarithmic form and exponential form.

Recall that exponential form ($x = b^y$) and logarithmic form ($y = \log_b x$) are equivalent.

Example:

(2) Convert to the other form in each case.

(a) $1{,}024 = 2^{10}$ Solution: $\log_2 1{,}024 = 10$

(b) $10^{-3} = 0.001$ Solution: $\log_{10} 0.001 = -3$ or $\log 0.001 = -3$

(c) $\ln e = 1$ Solution: $e^1 = e$

(d) $\log_3\left(\frac{1}{81}\right) = -4$ Solution: $3^{-4} = \frac{1}{81}$

(e) $a^r = t$ Solution: $\log_a t = r$

There are several "powerful" log rules that we will find very helpful for equation solving and applications. All can be proven using basic rules of exponents. Each rule is given below with some of the rationale for the rule and a few examples of its correct use. In each case, we make the following assumptions: $b > 0$, $b \neq 1$, $M > 0$, and $N > 0$.

Log Rule	**Example(s)**
1. $\log_b b = 1$ because $b^1 = b$	$\log_{50} 50 = 1$, $\log 10 = 1$, $\ln e = 1$
2. $\log_b 1 = 0$ because $b^0 = 1$	$\log_7 1 = 0$, $\ln 1 = 0$, $\log 1 = 0$
3. $\log_b b^n = n$ because $b^n = b^n$	$\log_8 8^{1.5} = 1.5$, $\ln e^{-3} = -3$, $\log 10^5 = 5$
4. $b^{\log_b n} = n$ because $\log_b n = \log_b n$	$4^{\log_4 6} = 6$, $10^{\log 9} = 9$, $e^{\ln 12} = 12$
5. $\log_b M \cdot N = \log_b M + \log_b N$ since $b^M \cdot b^N = b^{M+N}$	$\log_2 32 = \log_2 8 \cdot 4 = \log_2 8 + \log_2 4$ $\log 12 + \log 5 = \log 12 \cdot 5 = \log 60$
6. $\log_b \frac{M}{N} = \log_b M - \log_b N$ since $\frac{b^M}{b^N} = b^{M-N}$	$\log 100000 - \log 100 = \log \frac{100000}{100} = \log 1000$ $\ln 5 = \ln \frac{35}{7} = \ln 35 - \ln 7$
7. $\log_b M^p = p \cdot \log_b M$ since $\left(b^M\right)^p = b^{M \cdot p}$	$\log_2 4^3 = 3 \cdot \log_2 4$, $\log_{12} x^{10} = 10 \cdot \log_{12} x$

Examples: Use the log rules to find exact answers, whenever possible. Otherwise, use your calculator to approximate to 4 decimal places.

(3) $\log 10^8$ Solution: 8 (using rule 3)

(4) $\log_{2/3} 1$ Solution: 0 (using rule 2)

(5) $\log_2\left(\frac{1}{16}\right)$ Solution: $\log_2 1 - \log_2 16 = \log_2 1 - \log_2 2^4 = 0 - 4 = -4$ (using rules 6, 2, and 3)

(6) log 204,512 Solution: 5.3107 (using your trusty calculator!)

(7) log(–2) Solution: No solution (a domain issue for y = log x)

(8) ln 7.5 Solution: 2.0149 (again, using your trusty calculator!!)

(9) $5^{2\cdot\log_5 3}$ Solution: $5^{\log_5 3^2} = 5^{\log_5 9} = 9$ (using rules 7 and 4)

(10) $\log_{\sqrt{2}}\sqrt{2}$ Solution: 1 (using rule 1)

There is another extremely important log rule called the **change-of-base formula**, shown below.

Assuming that $b > 0$, $b \neq 1$, $M > 0$, $\log_b M = \frac{\log_a M}{\log_a b}$ for any base $a > 0$, $a \neq 1$.

Note: Common choices of base are 10 and e because they are the bases for LOG and LN keys on a scientific or graphing calculator.

Examples/Solutions: Use your calculator to approximate these to 4 decimal places.

(11) $\log_2 10 = \frac{\log 10}{\log 2} = \frac{1}{\log 2} \approx 3.3219$

(12) $\log_{23} 7 = \frac{\ln 7}{\ln 23} \approx 0.6206$

(13) $\log_{1/2} 40 = \frac{\log 40}{\log(1/2)} \approx -5.3219$

(14) $\log_5 25 = \frac{\log 25}{\log 5}$ or $\frac{\ln 25}{\ln 5} = 2$

This change-of-base formula enables you to use a graphing calculator to graph $y = \log_b x$ for any base b. For example, we saw the logarithmic function $y = \log_2 x$ graphed earlier using the form change technique ($x = 2^y$) and a table of values. Using the change-of-base formula, we may write $y = \frac{\log x}{\log 2}$ or $y = \frac{\ln x}{\ln 2}$. Entering either of these in the graphing calculator would produce the graph we found earlier. Similarly, in order to graph the logarithmic function $y = \log_5 x$ or $5^y = x$, we would use the change-of-base formula to write the equivalent function $y = \frac{\log x}{\log 5}$ or $y = \frac{\ln x}{\ln 5}$. All four of these forms describe the same function and curve.

In order to prepare for applications involving exponents and logarithms, we need to introduce some equation-solving techniques.

In each case below, we assume that $b > 0$, $b \neq 1$, $M > 0$, and $N > 0$.

Principles for Exponential and Logarithmic Equation Solving	
$y = \log_b x$ is equivalent to $b^y = x$	Changing from logarithmic form to exponential form, or vice versa, is a very common and powerful technique.
$\log_b M = \log_b N$ is equivalent to $M = N$	This is useful both directions — "dropping" the logs or taking the log of both sides of an equation
$b^M = b^N$ is equivalent to $M = N$	This is especially useful in the direction presented here — dropping the identical base, setting the exponents equal.

Examples: Solve for x. Be sure to check your solutions.

(15) $\log_x 5 = 2$

Solution: $\log_x 5 = 2$ is equivalent to $x^2 = 5$. Taking the square root of both sides of the equation, we have $x = \pm\sqrt{5}$. Since the base must be positive, the only solution is $x = \sqrt{5}$. You can check this with the equivalent form $\left[\left(\sqrt{5}\right)^2 = 5\right]$ or with the change-of-base formula $\left[\log_{\sqrt{5}} 5 = \frac{\log 5}{\log \sqrt{5}} = 2\right]$.

(16) $\log_2(4x) = -3$

Solution: This is equivalent to $2^{-3} = 4x$. Since $2^{-3} = \frac{1}{2^3}$ or $\frac{1}{8}$, we have $4x = \frac{1}{8}$. Multiplying both sides by $\frac{1}{4}$, we have $\frac{1}{4} \cdot 4x = \frac{1}{4} \cdot \frac{1}{8}$ or $x = \frac{1}{32}$. You can check this result with the change-of-base formula or using the equivalent form, as shown: $2^{-3} = 4\left(\frac{1}{32}\right) = \frac{1}{8}$.

(17) $2^x = 150$

Solution: Taking the log (or ln) of both sides, we have $\log 2^x = \log 150$. Using rule 7, we can bring the power in front of log 2: $x \cdot \log 2 = \log 150$. Then we can divide both sides of the equation by log 2 (or ln 2): $x = \frac{\log 150}{\log 2} \approx 7.2288$. This problem can also be handled using the equivalent form and the change-of-base formula. The check involves approximate values: $2^{7.2288} \approx 149.9980567 \approx 150$.

(18) $5^{2x} = \frac{1}{5}$

Solution: Since $\frac{1}{5} = 5^{-1}$, we have $5^{2x} = 5^{-1}$. Drop the base, setting the exponents equal, then solve for x: $2x = -1$ $x = -\frac{1}{2}$ To check, substitute: $5^{\left(2\cdot-\frac{1}{2}\right)} = 5^{-1} = \frac{1}{5}$.

Examples/Solutions: Solve for x. Be sure to check your solutions.

(19) $\ln (x + 1) - \ln x = \ln 4$ Solution: Combine the left-hand side using rule 6: $\ln \frac{x+1}{x} = \ln 4$.

Then "drop the logs", and solve the proportion using cross products and linear equation-solving techniques.

$\frac{x+1}{x} = \frac{4}{1}$ $\quad x + 1 = 4x \quad 1 = 3x \quad x = \frac{1}{3}$

The check is straightforward in the calculator:

$\ln\left(\frac{1}{3}+1\right) - \ln\left(\frac{1}{3}\right) \approx 1.3863$ and $\ln 4 \approx 1.3863$.

(20) $2 \log x = \log 25$ Solution: By rule 7, we may write $\log x^2 = \log 25$, then we can "drop the logs", and use the square root property. $x^2 = 25$

$x = \pm\sqrt{25} = \pm 5$ Because of the domain of logarithmic functions, x = 5 is the only solution, and the check is

$2 \log 5 \approx 1.3979$ and $\log 25 \approx 1.3979$.

Exercises:

1. Complete the chart, converting the given equation from one form to the other.

Logarithmic form	Exponential form
(a) $\log_3 81 = 4$	
(b) $\log 0.001 = -3$	
(c)	$7^5 = 16{,}807$
(d)	$e^{2.9957} \approx 20$
(e) $\log_5 1 = 0$	
(f)	$12^1 = 12$
(g) $\ln 39 \approx 3.6636$	

2. Evaluate. Give exact answers, whenever possible.

(a) $\log_2 64$ (b) $\log 100{,}000{,}000$ (c) $\ln\left(e^{-4}\right)$ (d) $\log (-10)$ (e) $\log_5 0.04$

3. Evaluate. Give exact answers, whenever possible.

(a) $\log_3 1$ (b) $\ln 0$ (c) $\log_4 (1/2)$ (d) $\log_{1/2} 4$ (e) $\log 0.000001$

4. (a) Complete the tables below for the given exponential and logarithmic functions.

$y = 3^x$

x	y
-3	
-2	
-1	
0	
1	
2	
3	
5	

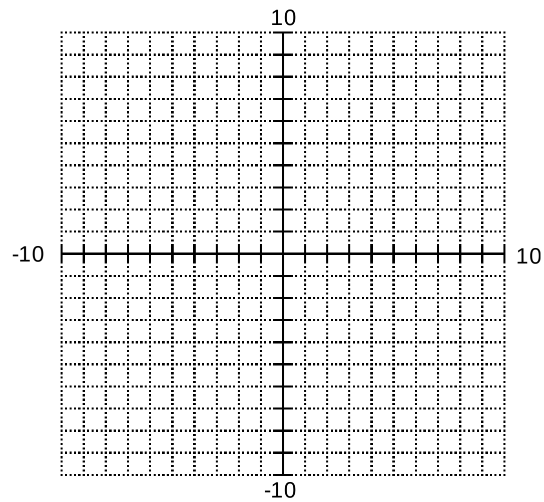
$y = \log_3 x$

x	y
	-3
	-2
	-1
	0
	1
	2
	3

(b) Graph these two functions along with y = x on the coordinate grid. Include any asymptote(s) and intercept(s).

(c) What do you notice about the tables in part (a)?

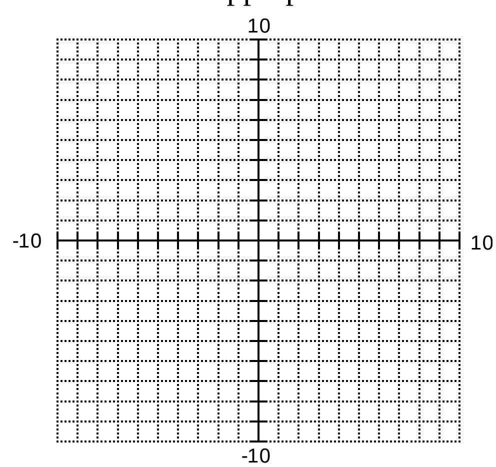
(d) What do you notice about the graphs?



5. For the logarithmic function given by $y = \log_2(x + 3) - 2$

(a) Complete the table below, and graph the function over an appropriate domain. Include any asymptote(s) and intercept(s).

x	y
-3	
-2	
-1	
0	
1	
2	
3	
5	



(b) How does this graph compare with $y = \log_2 x$ in shape and location on the coordinate grid? Refer to the graph of $y = \log_2 x$ that we developed earlier in this section.

6. Evaluate. Use your calculator and/or the log rules to approximate these to 4 decimal places.

(a) $\log 153$ (b) $\ln 44$ (c) $\log_7 28$

(d) $\log_5 1000$ (e) $\log_\pi e$ (f) $\log_7 3 \cdot \log_3 7$

(g) $8^{\log_8 1.25}$ (h) $2 \log_{5.789} 5.789^4$

7. Solve for x. Give exact answers, if possible. Otherwise, use your calculator to give approximate solutions correct to 4 decimal places.

(a) $\log_3 x = 10$ (b) $\log_x 10 = 3$ (c) $3^{2x-1} = 9$

(d) $\log_2(x+1) + \log_2(x-1) = \log_2 3$ (e) $5^x = 20$

8. Solve for x. Give exact answers, if possible. Otherwise, use your calculator to give approximate solutions correct to 4 decimal places.

(a) $3\log(x+1) = \log 8$ (b) $4^{-3x} = 72$ (c) $\log_4(5x-2) = \log_4(2x+7)$

(d) $\ln(2x-1) - \ln(x-2) = \ln 3$ (e) $10^x = 300$

9. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$.

10. Find the value of $\log_2 4 \cdot \log_4 6 \cdot \log_6 8 \cdot \log_8 16$.

11. Write each expression as a single logarithm.

(a) $2\log_3 u + 3\log_3 v$ (b) $3\log_3 u - \frac{1}{2}\log_3 v$

12. Write each expression as a sum and/or difference of logarithms. Express powers as factors.

(a) $\log_2(4x^2)$ (b) $\log_5\left(\frac{x}{25}\right)$ (c) $\log_7(x^7)$ (d) $\ln(ex)$