Logarithmic Applications

Here are some of the main ideas from the section on logarithmic functions:

- Exponential functions have the form $y = b^x$ (Recall that $b > 0$, $b \neq 1$).

- The inverse function for the exponential function is $x = b^y$ (This is called **exponential form** for a logarithmic function.) "Solving for y", an equivalent form is $y = \log_b x$ (This is called **logarithmic form** for a logarithmic function.)

- For any constant $b > 0$, $b \neq 1$, the equation $y = \log_b x$ defines a logarithmic function with base b and domain all $x > 0$.

There are several "powerful" log rules that we will find very helpful for equation solving and applications. All can be proven using basic rules of exponents. Each rule is given below with some of the rationale for the rule and a few examples of its correct use. In each case, we make the following assumptions: $b > 0$, $b \neq 1$, $M > 0$, and $N > 0$.

"Log Rules"

1. $\log_b b = 1$ because $b^1 = b$	5. $\log_b M \cdot N = \log_b M + \log_b N$ since $b^M \cdot b^N = b^{M+N}$
2. $\log_b 1 = 0$ because $b^0 = 1$	6. $\log_b \frac{M}{N} = \log_b M - \log_b N$ since $\frac{b^M}{b^N} = b^{M-N}$
3. $\log_b b^n = n$ because $b^n = b^n$	7. $\log_b M^p = p \cdot \log_b M$ since $\left(b^M\right)^p = b^{M \cdot p}$
4. $b^{\log_b n} = n$ because $\log_b n = \log_b n$	8. $\log_b M = \frac{\log_a M}{\log_a b}$

Principles for Exponential and Logarithmic Equation Solving

$y = \log_b x$ is equivalent to $b^y = x$	Changing from logarithmic form to exponential form, or vice versa, is a very common and powerful technique.
$\log_b M = \log_b N$ is equivalent to $M = N$	This is useful both directions — "dropping" the logs or taking the log of both sides of an equation
$b^M = b^N$ is equivalent to $M = N$	This is especially useful in the direction presented here — dropping the identical base, setting the exponents equal.

We mentioned that logarithmic and exponential functions have a variety of applications. These include the appreciation or depreciation of any investment, the pH of a chemical substance, the decibel scale for sound levels, magnitude of earthquakes, compound interest, population growth, and so on. Any of the types of problems in the exponential applications section have extension problems that can only be solved using logarithms.

Examples/Solutions:

(1) In chemistry, the acid potential of aqueous solutions is measured in terms of the pH scale. Tremendous swings in hydrogen ion concentration occur when acids or bases are mixed with water. The pH values range from negative values to numbers above 14. The pH of a substance is found using the formula $\text{pH} = -\log\left[\text{H}^+\right]$, where H^+ is the hydrogen ion concentration, measured in moles per liter. Pure water is neutral with a pH of 7; acids have a pH lower than 7; bases have a pH higher than 7.

Complete the chart.

Substance	**PH value**	$\left[\text{H}^+\right]$ **in mol/L**
(a) Tomatoes		$6.3 \cdot 10^{-5}$
(b) Lye	14.0	

Solutions: (a) For tomatoes, substitute $6.3 \cdot 10^{-5}$ for $\left[\text{H}^+\right]$ and evaluate the expression: $\text{pH} = -\log\left(6.3 \cdot 10^{-5}\right) = 4.2$. Tomatoes are acidic.

(b) For lye, start with $14 = -\log\left[\text{H}^+\right]$; multiply both sides by –1: $-14 = \log\left[\text{H}^+\right]$. Then we change from logarithmic form to exponential form: $\left[\text{H}^+\right] = 1 \cdot 10^{-14}$ mol/L (or 0.000 000 000 000 01 mol/L).

(2) The loudness L, in decibels (after Alexander Graham Bell), of a sound of intensity I is defined to be $L = 10 \log \frac{I}{I_0}$, where I_0 is the minimum intensity detectable by the human ear (such as the tick of a watch at 20 feet under quiet conditions). Complete the table.

Sound	**Intensity, I**	**Decibel Number**
(a) Rustle of leaves		20
(b) Jet 100 feet away at takeoff	$10^{14} \cdot I_0$	

Solutions: (a) Substitute 20 for L $\left[20 = 10 \log \frac{I}{I_0}\right]$, then divide both sides by 10 $\left[2 = \log \frac{I}{I_0}\right]$. Change from logarithmic form to exponential form $\left[10^2 = \frac{I}{I_0}\right]$, then multiply both sides of the equation by I_0, yielding the solution $I = 10^2 \cdot I_0$ or $100 \cdot I_0$.

(b) Here, you substitute the intensity, I, cancel the I_0 factors from the numerator and denominator, and then use rule 3 (or rules 7 and 1):

$$L = 10 \log \frac{10^{14} \cdot I_0}{I_0} = 10 \log 10^{14} = 10 \cdot 14 = 140 \text{ decibels}$$

(3) The magnitude R, measured on the Richter scale, of an earthquake of intensity I is defined as $R = \log\frac{I}{I_0}$, where I_0 is the threshold intensity for the weakest earthquake that can be recorded on a seismograph. If one earthquake is 10 times as intense as another, its magnitude on the Richter scale is 1 greater than the other. For instance, an earthquake that measures 7 is 10 times as intense as an earthquake of magnitude 6. If one earthquake is 100 times as intense as another, its magnitude on the Richter scale is 2 higher. Earthquake intensities can be interpreted as multiples of the minimum intensity I_0.
A magnitude 8 earthquake releases as much energy as detonating 6 million tons of TNT.

Complete the following table. In each case, round like the given values.

Earthquake	**Intensity, I**	**Richter Number, R**
(a) Philippine Islands Region (May 21, 2009)		5.9
(b) Kermodec Islands Region (May 16, 2009)	$10^{6.5} \cdot I_0$	

Solutions: (a) Substitute 5.9 for R $\left[5.9 = \log\frac{I}{I_0}\right]$, then change from logarithmic form to exponential form $\left[10^{5.9} = \frac{I}{I_0}\right]$, then multiply both sides of the equation by I_0, yielding the solution $I = 10^{5.9} \cdot I_0$ or $794{,}328 \cdot I_0$.

(b) Here, you substitute the intensity, I, cancel the I_0 factors from the numerator and denominator, and then use rule 3 (or rules 7 and 1) on page 271: $R = \log\frac{10^{6.5} \cdot I_0}{I_0} = \log 10^{6.5} = 6.5$

Note: Generally, to compare earthquakes, we may raise 10 to their difference in Richter scale magnitudes. Comparing these two earthquakes, we have $10^{6.5-5.9} \approx 3.98$, so the Kermodec Islands earthquake was approximately 4 times as intense at the Philippine Islands earthquake.

(4) When the compounding of interest on a checking or savings account is “continuous” (as opposed to quarterly, monthly or daily), the formula for the accumulated amount, A, of an investment (or loan) is given by the following formula, $A = P \cdot e^{r \cdot t}$, where P is the principal, r is the annual interest rate (expressed as a decimal), and t is the time in years. At 3% annual interest, compounded continuously, how long would it take for an initial investment of $500 to double in value? Round to the nearest tenth of a year.

Solution: First, substitute the known values, A = 2($500) = $1,000, P = $500, and r = 3% or 0.03, into the formula $\left[1{,}000 = 500\, e^{0.03 \cdot t}\right]$. Then divide both sides of this equation by 500 $\left[2 = e^{0.03 \cdot t}\right]$. Taking the natural log of both sides $\left[\ln 2 = \ln e^{0.03 \cdot t}\right]$ enables us to use rule 7 to bring the time variable out of the exponent $\left[\ln 2 = 0.03t \cdot \ln e\right]$. Since ln e = 1 (by rule 1), we divide both sides by 0.03 to find the final answer: $t = \dfrac{\ln 2}{0.03} \approx 23.1$ years.

[Note: This result generalizes to $t^* = \dfrac{\ln 2}{r}$, where t^* is the time it takes for an initial investment to double and r is the annual interest rate.]

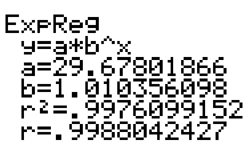
Because of the inverse relationship between exponential and logarithmic functions, there are a wide variety of applications involving logarithms. Now we will revisit an application involving exponential regression and Canada's population growth.

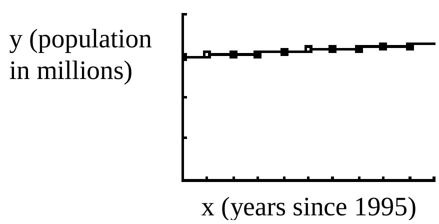
(5) The following data involves the mid-year population in Canada (on July 1 of each year). Source: Census Bureau

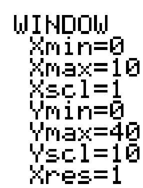
We input this data into the statistical lists of a TI calculator, with 0 for 1995, 1 for 1996, 2 for 1997, . . . , 8 for 2003, and 9 for 2004, along with the population figures given in the chart. As we saw before, the variable x represents the number of years since 1995, and y represents population in millions.

Year	**Population (in millions)**
1995	29.6
1996	30.0
1997	30.3
1998	30.6
1999	31.0
2000	31.3
2001	31.6
2002	31.9
2003	32.2
2004	32.5

We show the exponential regression analysis results, the scatter plot/curve, and the window inputs below.







As we see above, the TI exponential regression equation in $y = ab^x$ form is $y = 29.2780 \cdot 1.0104^x$. One of the common algebraic formulas used to predict a population after t years is $P = P_0 e^{k \cdot t}$, where P_0 is the initial amount, e is the number 2.71828 . . . , and k is a constant unique to the population in question (the growth rate). If we were to perform exponential regression using Excel, the result is $P = 29.2780 \cdot e^{0.0103x}$.

Comparing the two forms, notice that $P_0 = a$; these constants represent the initial population and y-intercept for the graph. There is also a direct relationship between the growth rate constant, k, and the value for b; if we let $e^k = 1.010356098$, we can simply take the natural logarithm of both sides, and use rules 7 and 1 (or rule 3), as shown:

$$\ln\left(e^k\right) = \ln 1.010356098$$

$$k \approx 0.0134698528$$

Exercises:

1. In chemistry, the acid potential of aqueous solutions is measured in terms of the pH scale. Tremendous swings in hydrogen ion concentration occur in water when acids or bases are mixed with water. These changes can be as big as 1×10^{14}. Since the pH is a logarithmic scale, every multiple of ten in H^+ concentration equals one unit on the scale. Physically, the pH is intended to tell what the acid potential is for the solution. The pH values range from negative values to numbers above 14. The pH of a substance is defined as $\text{pH} = -\log\left[H^+\right]$, where H^+ is the hydrogen ion concentration, measured in moles per liter. Pure water is neutral and has a pH of 7; acids have a pH lower than 7; bases have a pH higher than 7. Use the formula to complete the chart. Round like the given values.

Substance	**pH value**	$\left[H^+\right]$ **in mol/L**
(a) Eggs		$6.3 \cdot 10^{-5}$
(b) Apple	3.2	
(c) Household ammonia	11.6	
(d) Milk		$4.0 \cdot 10^{-7}$

2. The magnitude R, measured on the Richter scale, of an earthquake of intensity I is defined as $R = \log \frac{I}{I_0}$, where I_0 is the threshold intensity for the weakest earthquake that can be recorded on a seismograph. Complete the following table. In each case, round like the given values.

Earthquake	**Intensity, I**	**Richter Number, R**
(a) Chile, 1960		9.6
(b) Italy, 1980	$10^{7.85} \cdot I_0$	
(c) San Francisco, 1989 (during a MLB World Series)	$10^{6.9} \cdot I_0$	
(d) Kobe, Japan, 1995		6.8
(e) Indian Ocean, 2005		9.0
(f) Haiti, 2010		7.0
(g) Chile, 2010		8.8

3. The loudness L, in decibels (after Alexander Graham Bell), of a sound of intensity I is defined to be $L = 10 \log \frac{I}{I_0}$, where I_0 is the minimum intensity detectable by the human ear (such as the tick of a watch at 20 feet under quiet conditions). I_0 is measured to be 20 micropascals, or 0.02 mPa. (This is a very low pressure; it is 2 ten-billionths of an atmosphere. Nevertheless, this is about the limit of sensitivity of the human ear, in its most sensitive range of frequency. Usually this sensitivity is only found in rather young people or in people who have not been exposed to loud music or other loud noises.) If a sound is 10 times as intense as another, its loudness is 10 decibels greater; if a sound is 100 times as intense as another, its loudness is 20 decibels greater; and so on.

Complete the following table. Round like the given values.

Sound	**Intensity, I**	**Decibel Number**
(a) A whisper		18
(b) Pain threshold for a human ear	$10^9 \cdot I_0$	
(c) A crowded restaurant		80
(d) A library	$2{,}510 \cdot I_0$	

4. The formula $A = Pe^{r \cdot t}$ gives the accumulated amount (A) of an investment when P is the initial investment, r is the annual interest rate, and t is the time in years, assuming continuous compounding and no deposits or withdrawals.

(a) For an initial investment of $2,000, compounded continuously at a 7.5% annual interest rate, find to the nearest tenth of a year when this investment doubles in value.

(b) For an initial investment of $1,500, compounded continuously at a 4% annual interest rate, find to the nearest tenth of a year when this investment triples in value.

*5. The formula for the accumulated amount, A, of an investment (or loan) is given by the formula, $A = P(1+i)^n$, where P is the principal, i is the periodic interest rate, and n is the total number of interest periods. Compare these results with those found in #5 above.

(a) For an initial investment of $2,000, compounded monthly at a 7.5% annual interest rate, find to the nearest tenth of a year when this investment doubles in value.

(b) For an initial investment of $1,500, compounded quarterly at a 4% annual interest rate, find to the nearest tenth of a year when this investment triples in value.

*6. A strain of E-coli Beu 397-recA441 is placed into a petri dish at 30°C and allowed to grow. The following data are collected. Theory states that the number of bacteria in the petri dish will initially grow according to the law of uninhibited growth (modeled by an exponential function). The population is measured using an optical device in which the amount of light that passes through the petri dish is measured.

Time (hours), t	0	2.5	3.5	4.5	6
Population, N	0.09	0.18	0.26	0.35	0.50

(a) Find the regression equation of the exponential function which best fits the bacteria population trend (in the form $N = N_0 \cdot e^{kt}$). Round coefficients to 8 decimal places.

(b) Use your regression equation to predict the population at 5 hours. Round to the nearest hundredth.

(c) Use your regression equation to predict when the population will reach 0.75. Round to the nearest tenth of an hour.

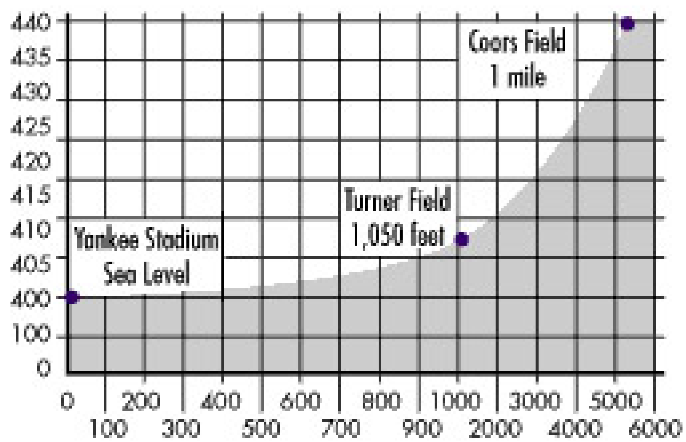
*7. Refer to the given world population growth data. Source: United Nations
Let 1900 = Year 0, and let t represent the number of years since 1900.

YEAR, t	**0**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**
P (billions)	1.65	1.75	1.86	2.07	2.30	2.52	3.02	3.70	4.44	5.27	6.06

(a) Find the regression equation of the exponential function which best fits the world population trend. Round coefficients to 4 decimal places.

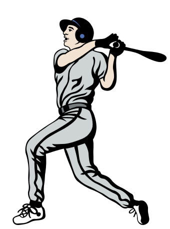
(b) Experts estimate that Earth's natural resources are sufficient for up to around 40 billion people. To the nearest year, predict when world population will reach 40 billion.

8. Coors Field in Denver, Colorado is home of Major League Baseball's Colorado Rockies. In 1999, the team belted 303 home runs, the most ever in a season at one ballpark. One of the determining factors is the altitude of the ballpark. Experts estimate that the ball travels 9 percent farther at 5,280 feet than at sea level. As the graph indicates, a home run hit 400 feet in sea-level Yankee Stadium would travel about 408 feet in Atlanta (second-highest in the majors at 1,050 feet) and as far as 440 feet in the "Mile High City". The wind can easily play a much greater role than altitude in turning fly balls into home runs. The same 400-foot shot, with a 10-mph wind at the hitter's back, can turn into a 430-foot blast. Another important effect of altitude on baseball is the influence thinner air has on pitching. In general, curve balls will be a little less snappy, and fastballs will go a little faster due to the decrease in resistance the thinner air provides.



(a) Find the regression equation of the exponential function which best fits the 3 points given in the graph above, i.e., (0, 400), (1050, 408), and (5,280, 440). Use the form $y = a \cdot b^x$, where x represents altitude in feet and y represents distance the ball travels in feet). Round constants to 8 decimal places.

(b) Use your regression equation to predict the distance the ball would travel at altitudes of 500 ft __________ and 3,542 ft __________ (assuming it would travel 400 ft at sea level).



(c) Use your regression equation to predict the altitude of the stadium where the ball would travel 420 feet (assuming it would travel 400 ft at sea level). Round to the nearest foot.