Exponential Applications

There are many, many applications of exponential functions, including compound interest, population growth, and radioactive decay. Also, several scales involve exponential functions (and its inverse, logarithmic functions), including pH, sound and Richter scale readings. We will consider several examples.

Examples:

(1) The Salary Problem:

Thomas earns $1 on the first day of a job. The second day, he is paid $2, the third day $4, the fourth day $8, and so on. Each day, his salary is twice that of the previous day. He plans to stay on the job for 21 days and wishes to know what his total earnings will be. Find an easier way to determine his total salary without adding up all twenty-one days' earnings.

Solution:

Clearly the daily wage amounts are powers of 2. Since $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and $2^3 = 8$, it is reasonable to conclude that on day 21, Thomas will be paid 2^{20} dollars. In general, he is paid 2^{n-1}, where n is the day number. The chart below shows the complete table.

Day Number, n	Thomas' Wages ($)
1	1
2	2
3	4
4	8
5	16
6	32
7	64
8	128
9	256
10	512
11	1,024
12	2,048
13	4,096
14	8,192
15	16,384
16	32,768
17	65,536
18	131,072
19	262,144
20	524,288
21	1,048,576

To tackle the cumulative total question, we notice that the cumulative totals follow a similar pattern. Each value is one less than a power of 2.

For instance, on day 1, Thomas received $2^1 - 1 = \$1$. On day 2, he received $2^2 - 1 = \$3$, and on day 3, he received $2^3 - 1 = \$7$. So, we conclude that on day 21, Thomas has received a total of $2^{21} - 1$ or \$2,097,151.

Day Number, n	Thomas' Wages ($)	For a total of . . .
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
6	32	63
7	64	127
8	128	255
9	256	511
10	512	1,023
11	1,024	2,047
12	2,048	4,095
13	4,096	8,191
14	8,192	16,383
15	16,384	32,767
16	32,768	65,535
17	65,536	131,071
18	131,072	262,143
19	262,144	524,287
20	524,288	1,048,575
21	1,048,576	2,097,151

To say the least, this is a surprisingly high amount (\$2,097,151 for 21 days work), and Thomas is quite pleased with the arrangement!

The formula for the accumulated amount of an investment (or loan) is given by the formula, $A = P(1 + i)^n$, where P is the principal, i is the periodic interest rate, and n is the total number of interest periods. It is customary to round the amount A <u>down</u> to the nearest hundredth's place.

To find i, we divide the annual interest rate (r) by the number of compoundings (m) for a year.

To find n, we often multiply the number of compoundings in a year (m) by the number of years (t) the investment is allowed to grow.

In general, we may use the formulas, $i = \frac{r}{m}$ and $n = m \cdot t$. Also, the amount of interest (I) accumulated is how much the accumulated amount exceeds the original principle: I = A – P.

Common numbers of compoundings for a year include:

1 .... for compounded “annually” (once a year)
2 .... for “semiannually” (every 6 months)
4 .... for “quarterly” (every 3 months)
12 .... for “monthly” (once a month)
24 .... for “bimonthly” (twice a month)
26 .... for “biweekly” (every 2 weeks)
52 .... for “weekly” (once a week)
360* .... for “daily” (every day)

*Most banks use a 360-day “year”.

(2) Find the accumulated amount on an investment of \$2,000 at 9% annual interest, compounded quarterly over 8 years.

Solution: $$A = 2000\left(1+\frac{0.09}{4}\right)^{4\cdot 8} = 2000\left(1+\frac{0.09}{4}\right)^{32} \approx \$4{,}076.20$$

(3) Find the accumulated amount on an investment of \$2,000 at 9% annual interest, compounded daily over 8 years.

Solution: There are $360 \cdot 8$ or 2,880 compoundings, and

$$A = 2000\left(1+\frac{0.09}{360}\right)^{2880} \approx \$4{,}076.20$$

(4) Find the accumulated amount on an investment of \$4,000 at 6% annual interest, compounded monthly over 5 years.

Solution: There are $5 \cdot 12$ or 60 compoundings, and

$A = 4000\left(1+\frac{0.06}{12}\right)^{60} \approx \$5{,}395.40$. In this case, the total interest is

I = A – P = \$5,395.40 – \$4,000 = \$1,395.40

As the number of compoundings (m) increases toward infinity, we saw in the previous section that the quantity $\left(1 + \frac{1}{m}\right)^{m}$ approaches the number e. As a direct result, if interest is **compounded “continuously”**, we use a new formula, $A = Pe^{r\cdot t}$, where P is the principal, r is the annual interest rate, and t is the time in years.

(5) Find the accumulated amount if \$2,000 is invested at 6.5% compounded continuously for 7 years.

Solution: $A = Pe^{rt} = 2000e^{0.065\,7} \approx \$3{,}152.35$

There are 4 variables that affect the accumulated amount, A — the annual interest rate (r), the principal (P), the time in years (t), and the number of compoundings per year (m). Here, we will focus on one of these 4 cases in more depth.

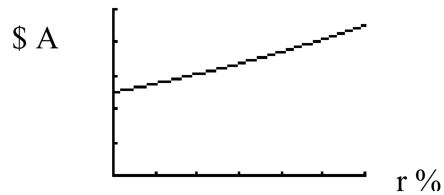
(6) Find the accumulated amount on an investment of $2,500 at r% annual interest, compounded quarterly over 5 years.

(a) Let r be 0, 2, 4, 6, and 8. Make and complete a table showing these r values with the corresponding A values.

Annual interest rate (r %)	Accumulated amount ($A)
0	2,500.00
2	2,762.23
4	3,050.47
6	3,367.13
8	3,714.86

Each calculation is done just like in Examples 2–4. Remember to round down to the nearest hundredth.

(b) Sketch the graph of r vs. A. [A is the dependent variable.]



Our scale on the r axis is 2%, and our scale on the A axis is $1,000.

(c) What type of function is it? (linear, polynomial, rational, exponential, logarithmic, radical, or a combination of these)

This is a polynomial function with degree 20, resembling a quadratic function.

(d) Describe features of the functions using the following terms: intercept, slope, vertex, asymptote, maximum, minimum, symmetry (whichever applies).

The y-intercept is (0, 2500). This is also the minimum point for the curve. No other features apply to this function.

(e) As r increases, A ______________________________ (Fill in the blank with: increases steadily, increases by more and more, increases by less and less, decreases steadily, decreases by more and more, or decreases by less and less.)

A increases by more and more. We see this both by the graph and by the table.

(f) Write the specific equation relating r and A. (A as a function of r)

$$A = P(1 + i)^n = 2{,}500\left(1 + \frac{r}{4}\right)^{20}$$

(7) Another application of exponential functions is uninhibited or exponential growth (and decay). The following data involves the mid-year population in Canada (on July 1 each year).

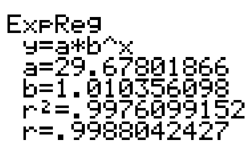
Input this data into the statistical lists of a TI graphing calculator, with 0 for 1995, 1 for 1996, 2 for 1997, . . . , 8 for 2003, and 9 for 2004, along with the population figures given in the chart.

Here, the variable x represents the number of years since 1995, and y represents population in millions.

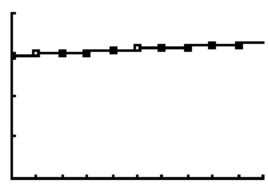
Then have the calculator perform exponential regression analysis on the data set using the ExpReg command in the STAT CALC menu.

Year	**Population (in millions)**
1995	29.6
1996	30.0
1997	30.3
1998	30.6
1999	31.0
2000	31.3
2001	31.6
2002	31.9
2003	32.2
2004	32.5

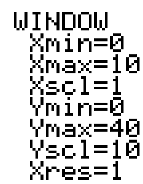
We show the exponential regression analysis results, the scatter plot/curve, and the window inputs below.



y (population in millions)



x (years since 1995)



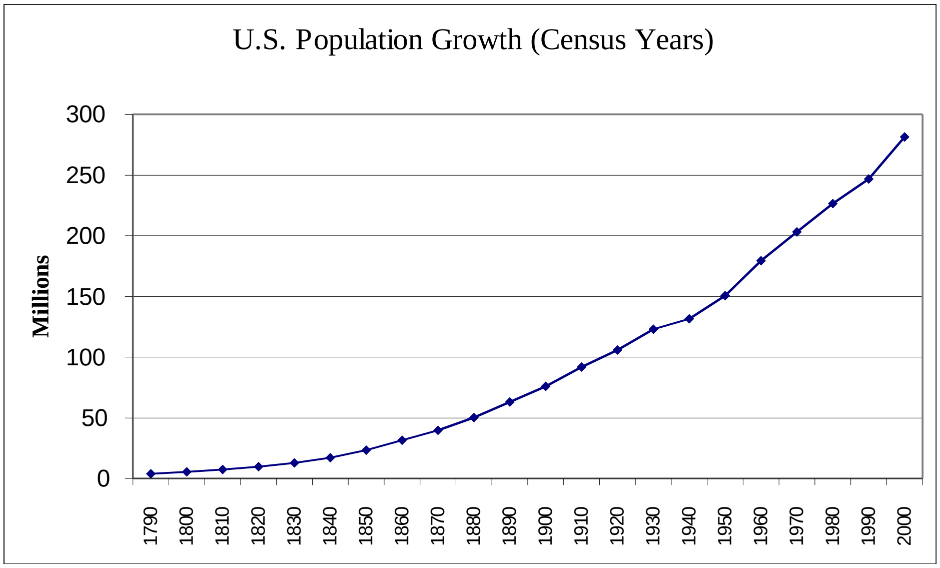
As we see above, the TI calculator exponential regression equation in $y = ab^x$ form is $y = 29.2780 \cdot 1.0104^x$. One of the common algebraic formulas used to predict a population after t years is $P = P_0 e^{k \cdot t}$, where P_0 is the initial amount, e is the number 2.71828 . . . , and k is a constant unique to the population in question. The constant k is the growth rate. If we were to perform exponential regression using Excel, the result is $y = 29.2780 \cdot e^{0.0103x}$. In both cases, x and t are interchangeable, and y and P are also interchangeable.

Comparing the two forms, you will notice that $P_0 = a$; these constants represent the initial population and y-intercept for the graph. You will also notice a direct relationship between the growth rate constant, k, and the value for b. In the next section, we will develop logarithm rules that can help us go back and forth between these two forms.

We could predict Canada's population in 2010 (fifteen years after 1995) by substituting 15 for x in the formula, as we show below: $y = 29.2780 \cdot 1.0104^{15} \approx 34.2$ million. Using the equivalent formula, we have $y = 29.2780 \cdot e^{0.010315} \approx 34.2$ million.

Although exponential functions typically model population growth, Canada's steady growth is very close to a linear relationship.

The United States, on the other hand, has had growth more clearly resembling the exponential function, as we see in the graph below.



Use the given broken line graph to approximate the U.S. population in the following years.

(a) 1900 (b) 1960 (c) 2000

Estimates: (a) 75 million (b) 180 million (c) 280 million

You could also extend the curve and offer predictions for years beyond the year 2000. Given the actual data for these census years, you could also perform exponential regression analysis and use the equation to predict the population for any given year more precisely.

(8) Richter Scale readings (for the intensity of an earthquake):

The Richter scale is used to measure and compare the strength of earthquakes. Each increase of one on the Richter scale corresponds to a ten-fold increase in intensity. In other words, an earthquake that registers 7 on the Richter scale is ten times as intense as an earthquake that registers 6 and one hundred times as intense as an earthquake that registers 5. The table below gives the impact of earthquakes of varying intensities.

Richter Number	**Impact**
1	Only detectable by seismograph
2	Hanging lamps sway
3	Can be felt
4	Glass breaks, buildings shake
5	Furniture collapses
6	Wooden houses damaged
7	Buildings collapse
8	Catastrophic damage

To compare the strengths or intensities of two earthquakes, a simple formula can be used: $10^{R_1 - R_2}$, where R_1 and R_2 represent the two Richter scale readings.

For example, in 1906, an earthquake measuring 6.9 wreaked havoc in San Francisco, causing property damage and fires, plus hundreds of people died. The Mexico City earthquake of 1985 registered 8.1 on the Richter scale. The earthquake in the Indian Ocean in early 2005 that triggered the tsunami impacting much of Southeast Asia registered 9.0 on the Richter scale. How do their intensities of these earthquakes compare?

Solution: Since $10^{8.1-6.9} \approx 15.8$, the Mexico City earthquake was nearly 16 times as intense as the San Francisco earthquake.

Since $10^{9.0-6.9} \approx 125.9$, the tsunami earthquake was nearly 126 times as intense as the San Francisco earthquake, and since $10^{9.0-8.1} \approx 7.9$, the tsunami earthquake was nearly 8 times as intense as the Mexico City earthquake.

Exercises:

1. The formula for the accumulated amount, A, of an investment (or loan) is given by the formula, $A = P(1 + i)^n$, where P is the principal, i is the periodic interest rate, and n is the total number of interest periods. In each graph, consider the y-intercept and clearly label the axes, with the dependent variable A on the vertical axis each time. Round all accumulated amounts down to the nearest hundredth's place.

   (a) Find the accumulated amount on an investment of $P at 9% annual interest, compounded quarterly over 8 years. Let P be $1,000, $2,000, $3,000, $4,000, and $5,000. Sketch the graph of P vs. A.

   (b) Find the accumulated amount on an investment of $2,500 at r% annual interest, compounded quarterly over 5 years. Let r be 2, 4, 6, 8, and 10. Sketch the graph of r vs. A.

   (c) Find the accumulated amount on an investment of $2,500 at 7% annual interest, compounded m times each year over 5 years. Let m be 1, 4, 12, 52, and 360. Sketch the graph of m vs. A.

   (d) Find the accumulated amount on an investment of $1,000 at 6% annual interest, compounded monthly over t years. Let t be 5, 10, 15, 20, and 40. Sketch the graph of t vs. A.

2. Refer to problem 1. For each of the 4 cases, what type of function is involved? (linear, polynomial, rational, exponential, logarithmic, radical, or a combination of these). Also, write the specific equation relating the independent variable and A. (A as a function of the independent variable)

3. Refer to problem 1, part c. As m approaches infinity (compounding "continuously"), use the formula, $A = Pe^{r \cdot t}$ (where P is the principal, r is the annual interest rate, and t is the time in years) to determine the value that A approaches? Note: This value is an asymptotic number for the graph.

4. In October 1999, an earthquake measuring 7.0 occurred in a sparsely populated part of the Mojave desert in California. The earthquake which rocked western Turkey in August 1999 registered 7.4 on the Richter scale. An earthquake registering 7.6 shook Taiwan in September 1999, shortly after the disaster in Turkey. And earlier in 1999, Colombia experienced an earthquake registering 5.8 near Bogotá.

Compare the intensities of the earthquakes in

(a) California and Taiwan

(b) Colombia and California

(c) Turkey and Taiwan

5. World demand for timber is increasing exponentially. The demand N, in billions of cubic feet, purchased is given by $N = 46.6 \cdot 1.108^t$, where t is the number of years since 1981. *(Source: U.N. Food and Agricultural Organization, American Forest and Paper Association)*

(a) Use the equation to predict the demand for timber in 1985, 1997, and 2005. Round to the nearest tenth of a billion cubic feet.

(b) Graph the function.

(c) After how many years will the demand for timber be 93.4 billion cubic feet? Round to the nearest year.

6. Refer to the given United States population growth data. Source: Census Bureau

YEAR	**1910**	**1920**	**1930**	**1940**	**1950**	**1960**	**1970**	**1980**	**1990**	**2000**
P (millions)	92.4	106.5	123.2	132.1	152.3	180.7	205.1	227.7	250.1	282.4

(a) Make a scatter plot for the given table. Place year on the horizontal axis, and U.S. population on the vertical axis. Let x represent the number of years since 1900.

(b) Find the regression equation of the exponential function which best fits the U.S. population trend (in the form $y = a \cdot b^x$). Round constants to 4 decimal places.

(c) Find the correlation coefficient to 4 decimal place precision.

(d) Use your regression equation to predict the population of the U.S. in 2010. Round to the nearest tenth of a million people.

7. The amount of the monthly payment (M) necessary to repay a loan with interest depends on several things: the amount borrowed (P, for principle), the monthly interest rate (i), and the total number of months (n). The formula for calculating the monthly payment is:

$$M = \frac{P\,i\,(1+i)^n}{(1+i)^n - 1}$$

Note: It is customary for banks and other lending institutions to round up to the nearest hundredth's place.

You need to borrow $10,000 to buy a car, and you determine that you can afford monthly payments of $250. The bank offers three choices: a 3-year loan at 7% APR, a 4-year loan at 7.5% APR, or a 5-year loan at 8% APR.

(a) Find the monthly payment involved with each choice of interest rate.

7% __________ 7.5% __________ 8% __________

(b) Which loan(s) fit your budget?

(c) For the loan(s) that fit(s) within your budget, calculate the total interest you will pay.

(d) Which loan would you choose? Explain.

*8. Refer to the U.S. population data and exponential regression function you found in Exercise 7. Predict the population in the year of your birth. Then go to the website http://www.census.gov/population/www/popclockus.html and find the actual population for your birth year. Notice that all data is approximate (and logistically difficult to find), and each value given is for July 1 of that year.

9. Which of the following situations would yield the larger amount in 1 year?

(a) 6% compounded quarterly or $6\frac{1}{4}$% compounded annually

(b) 5% compounded semiannually or 4.9% compounded daily

[Hint: Use an initial principal of $1,000 or $100 for each case, then compare the accumulated amounts.]

10. The number of rabbits in a certain population doubles every month, and there are 20 rabbits present initially. How many rabbits are present after

(a) 1 year?

(b) 20 months?

(c) 2 years?

11. A town has a population of 50,000 and is increasing at a rate of 2.5% each year. Using the equation, $P(t) = 50{,}000 \cdot 1.025^t$, estimate to the nearest year when the population of the town will reach 75,000.

*12. Al wanted his father to give him an allowance of $1 a week, but his father refused to go higher than 50¢.

Al thought about the situation and said: “Tell you what, Dad. Today is the first of April. You give me a penny today. Tomorrow give me two pennies. The next day, give me four pennies. Each day give me twice as many pennies as you did the day before.”

“For how long?” asked Dad, a little warily.

“Just for the month of April,” said Al. “Then I won’t ask for any more money for the rest of my life.”

“Okay,” Dad said quickly. “It’s a deal!”

How much will Dad need to pay in all during the month?

*13. According to the U.S. Census Bureau, the growth rate of the world’s population in 1997 was k = 1.33% = 0.0133. The population of the world in 1997 was 5,840,445,216. Letting t = 0 represent 1997, use the uninhibited growth (or exponential function) model to predict the world’s population in 2010.

*14. The future value S of an **ordinary annuity** consisting of n equal payments of R dollars, each with interest rate i per period (payment interval), is given by $S = R\dfrac{(1+i)^n - 1}{i}$.
Suppose you make monthly $25 deposits into a retirement account that pays an APR of 9% or 0.75% monthly. What is the value S of this annuity at the end of 40 years? How much of this is principal (your deposits)?

*15. Sketch the graph of $y = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ using these window settings: [–3, 3, 1, –0.2, 0.6. 0.1].
This is the **standard normal curve** and is extremely useful in a variety of statistical applications.