Function Transformations

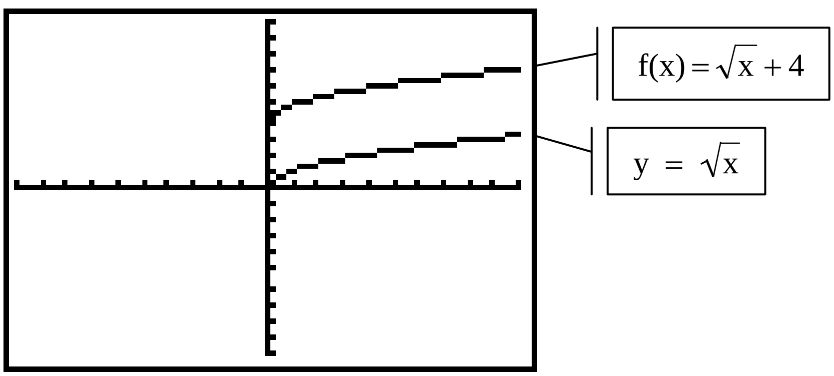
At this stage, if you were asked to graph $y = x$, $y = x^2$, $y = \sqrt{x}$, $y = |x|$, $y = x^3$, $y = \sqrt[3]{x}$, and $y = \frac{1}{x}$, you'd recognize them. If not, refer back to the "Library of Functions" section. Sometimes we are asked to graph a function that is "almost" the same as one of the building block functions. Here, we will consider several techniques for graphing such related functions. Collectively, these techniques are referred to as **transformations**.

TI-83 note: We will be using a variety of window settings, but unless otherwise indicated, the scales for both the horizontal and vertical axes will be 1.

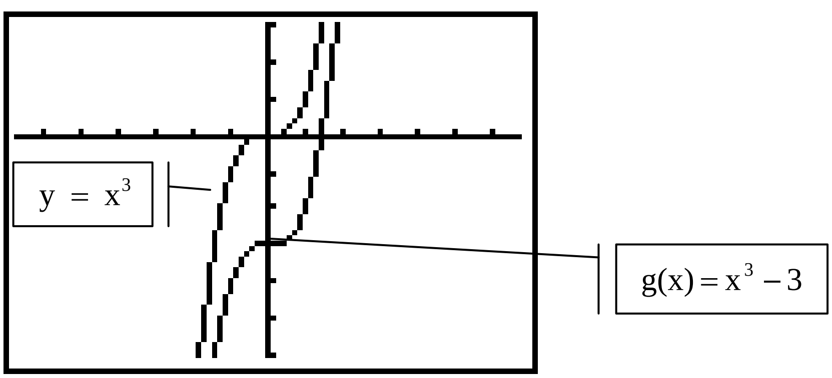
Vertical Shifts:

Consider the graphs of the functions $f(x) = \sqrt{x} + 4$ and $g(x) = x^3 - 3$. These are directly related to the square root function and to the cubing function "building blocks".

In the case of f, the endpoint is (0, 4), and evidently, this is the graph of $y = \sqrt{x}$ shifted up 4 units. Referring to the summary, in this case, k is 4.



In the case of g, the inflection point (and y-intercept) is (0, -3), and this looks just like $y = x^3$ shifted down 3 units. Referring to the summary, in this case, k is -3.

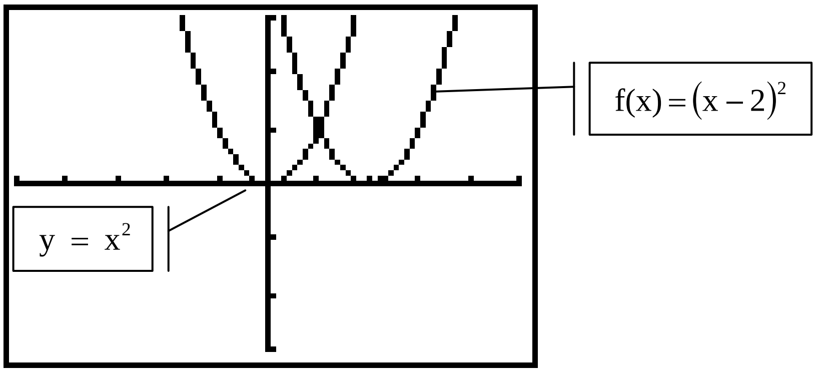


If a real number k is added to the right side of a function y = f(x), the graph of the new function y = f(x) + k is the graph of the original function shifted vertically up if k > 0 or down if k < 0.

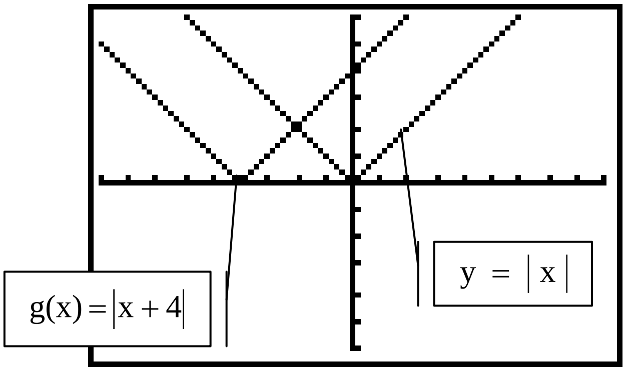
Horizontal Shifts:

Consider the graphs of the functions $f(x) = (x - 2)^2$ and $g(x) = |x + 4|$. These are directly related to the squaring function and to the absolute value function "building blocks".

In the case of f, the vertex moves from the origin to (2, 0) and evidently, this is the graph of $y = x^2$ shifted right 2 units. Referring to the summary, in this case, h is 2.



In the case of g, the minimum point (and x-intercept) is (-4, 0), and this looks just like $y = |x|$ shifted left 4 units. Referring to the summary, in this case, h is –4: $y = |x - (-4)| = |x + 4|$.

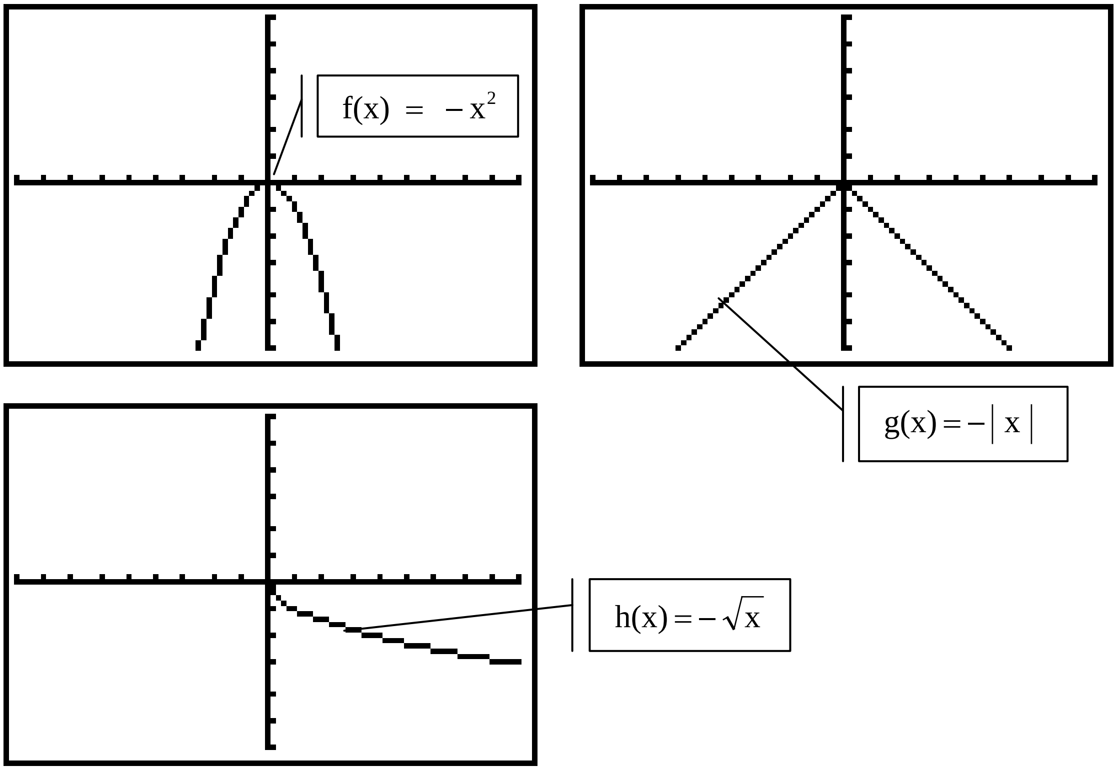


> If the argument x of a function is replaced by x – h, with h any real number, the graph of the new function y = f(x – h) is the graph of f shifted horizontally left if h < 0 or right if h > 0.

Reflections over the x-axis:

Consider the graphs of the functions $f(x) = -x^2$, $g(x) = -|x|$, and $h(x) = -\sqrt{x}$.

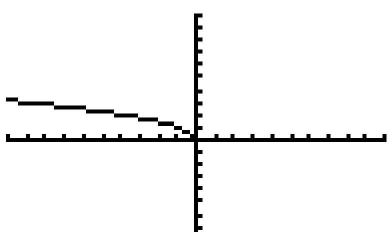
In the case of f, the building block graph of $y = x^2$ is reflected over the x-axis.
In the case of g, the building block graph of $y = |x|$ is reflected over the x-axis.
In the case of h, the building block graph of $y = \sqrt{x}$ is reflected over the x-axis.



When the right side of the function y = f(x) is multiplied by -1, the graph of the new function y = -f(x) is the reflection about the x-axis of the graph of y = f(x).

Reflections over the y-axis:

Consider the graph of the function $f(x) = \sqrt{-x}$.



This is the graph of $y = \sqrt{x}$ reflected over the y-axis. The endpoint is still (0, 0), but the domain has changed to $x \leq 0$.

This principle applies to all functions. For instance, the graph of $y = \sqrt[3]{-x}$ is the graph of the building block $y = \sqrt[3]{x}$ reflected over the x-axis.

One other note — The graphs of several building block functions are symmetric with respect to the y-axis, and the reflection over the y-axis would produce an identical graph. For instance, the graphs of $y = f(x) = x^2$ and $y = f(-x) = (-x)^2 = x^2$ are truly the same, point by point.

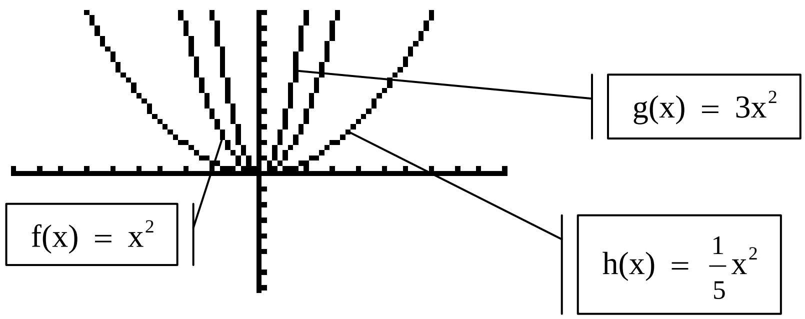
When the graph of the function y = f(x) is known, the graph of the new function y = f(-x) is the reflection about the y-axis of the graph of the function y = f(x).

Vertical Stretches and Compressions:

Consider the graphs of $f(x) = x^2$, $g(x) = 3x^2$, and $h(x) = \frac{1}{5}x^2$. The function f is the "building block" squaring function. Compare the other two graphs with f.

The graph of g is vertically stretched by a factor of 3; in other words, for a given x value, the y value is 3 times as far from the x-axis. In general, g(x) = 3 f(x).

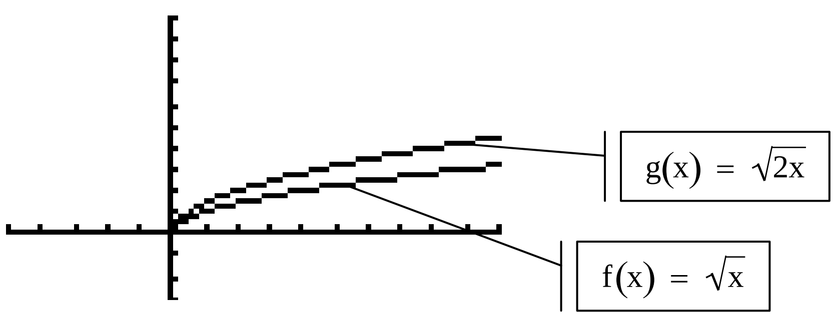
The graph of h is vertically compressed by a factor of $\frac{1}{5}$; in other words, for a given x value, the y value is one-fifth as far from the x-axis. In general, $h(x) = \frac{1}{5} f(x)$.



When the right side of the function y = f(x) is multiplied by a positive number a, the graph of the new function y = a f(x) is obtained by multiplying each y-coordinate on the graph of y = f(x) by a. The new graph is a vertically compressed version (if 0 < a < 1) or a vertically stretched version (if a > 1) of the graph of y = f(x).

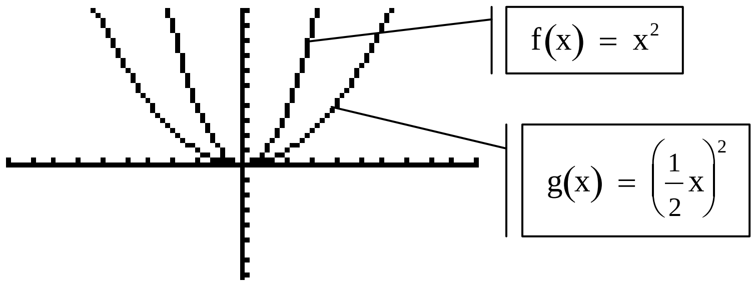
Horizontal Stretches and Compressions:

Consider the graphs of $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2x}$.



The graph of g is a horizontally compressed version of f by a factor of 2; in other words, for a given y value, the x value is one-half as far from the y-axis. In general, g(x) = f(2x). Notice the reciprocal effect of the value with the x.

Now consider the graphs of $f(x) = x^2$ and $g(x) = \left(\frac{1}{2}x\right)^2$.



The graph of g is a horizontally stretched version of f by a factor of 2; in other words, for a given y value, the x value is twice as far from the y-axis. In general, g(2x) = f(x). Notice again the reciprocal effect of the value with the x.

If the argument x of a function y = f(x) is multiplied by a positive number a, the graph of the new function y = f(ax) is obtained by multiplying each x-coordinate on the graph of y = f(x) by $\frac{1}{a}$. The new graph is a horizontally compressed version (if a > 1) or a horizontally stretched version (if 0 < a < 1) of the graph of y = f(x).

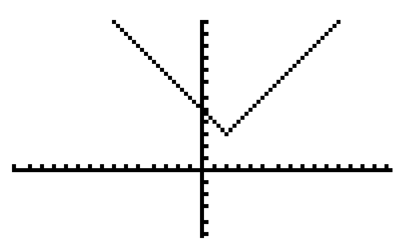
You may notice that g(x) above may be rewritten as $g(x) = \frac{1}{4}x^2$. Thus, using the form $g(x) = \left(\frac{1}{2}x\right)^2$, we can think of g as a horizontally stretched version of f by a factor of 2. Using this other form, we can think of g as a vertically compressed version of f by a factor of $\frac{1}{4}$. For many functions, a vertical stretch is a horizontal compression, and a horizontal stretch is a vertical compression.

Let's consider a few examples, "putting it all together".

(1) Graph the function given by $y = |x - 2| + 3$.

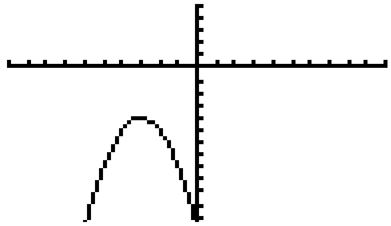
This graph should look exactly like $y = |x|$ horizontally shifted right 2 units (with h = 2) and vertically shifted up 3 units (with k = 3). The minimum point would then be (2, 3).

Points on the graph include (0, 5), (1, 4), (2, 3), (3, 4), and (4, 5).



(2) Graph the function given by $f(x) = -(x + 3)^2 - 4$.

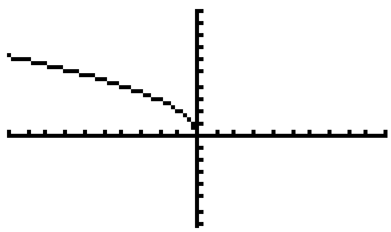
This graph should look like $y = x^2$ reflected over the x-axis (because of the negative sign in front of the parentheses), then horizontally shifted left 3 units and vertically shifted down 4 units. The maximum point, or vertex, of this parabola is (-3, -4). In this case, h is -3 and k is -4. Notice the symmetry in the table about the vertex.



x	y
0	-13
-1	-8
-2	-5
-3	-4
-4	-5

(3) Graph the function given by $g(x) = 2\sqrt{-x}$.

This graph should look like $y = \sqrt{x}$ reflected over the y-axis and then vertically stretched by a factor of 2. The endpoint is still the origin. You'll see a similar function, $y = \sqrt{-x}$, in the same window, earlier in this document.



x	y
0	0
-1	2
-4	4
-9	6

(4) Write an equation whose graph can be obtained from $y = \sqrt{x}$ by

(a) Shifting every point on the graph left 2 units and down 3 units.

Letting h = -2 and k = -3, we have $y = \sqrt{x - (-2)} - 3 = \sqrt{x + 2} - 3$.

(b) Shifting every point on the graph right 5 units.

Substituting h = 5, we have $y = \sqrt{x - 5}$.

(5) Write an equation whose graph can be obtained from $y = |x|$ by

(a) Shifting every point on the graph up 4 units.

Substituting k = 4, we have $y = |x| + 4$.

(b) Reflecting over the x-axis.

Placing a negative sign in front of the absolute value symbol, we have $y = -|x|$.

(6) Write an equation whose graph can be obtained from $y = \sqrt{x}$ by

(a) Reflecting over the y-axis and vertically stretching the graph by a factor of 3.

Placing a negative sign with the x and a 3 in front of radical sign, we have $y = 3\sqrt{-x}$.

(b) Horizontally compressing the graph by a factor of 3.

Placing a 3 under the radical sign with the x, we have $y = \sqrt{3x}$.

Exercises:

1. Write the equation of the function described below.

(a) The same as the graph of $y = x^3$, but shifted horizontally right 5 units

(b) The same as the graph of $y = |x|$, but shifted vertically up 3 units

(c) The same as the graph of $y = \sqrt{x}$, but shifted left horizontally 2 units and reflected over the x-axis

(d) The same as the graph of $y = x^2$, but vertically stretched by a factor of 4

(e) The same as the graph of $y = \sqrt{x}$, but horizontally stretched by a factor of 3.

2. Describe in a sentence the relationship between the graphs of $y = x^2$ and

(a) $y = -(x + 3)^2$

(b) $y = \frac{1}{2}x^2 - 4$

(c) $y = 2(x - 3)^2 + 4$

3. Find the equation of the function whose graph is the graph of $y = x^3$, but is:

(a) Shifted up 6 units

(b) Reflected about the y-axis

(c) Reflected about the x-axis

(d) Shifted right 4 units

(e) Vertically stretched by a factor of 1.5

4. If (3, 0) is on the graph of y = f(x), which of the following points must be on the graph of y = -f(x)? Multiple choice.

(a) (3, 0) (b) (0, 3) (c) (0, -3) (d) (-3, 0)

5. If (0, 3) is on the graph of y = f(x), which of the following points must be on the graph of y = 2f(x)? Multiple choice.

(a) (0, 3) (b) (0, 6) (c) (0, 2) (d) (6, 0)

6. If (3, 0) is on the graph of y = f(x), which of the following points must be on the graph of y = f(-x)? Multiple choice.

(a) (0, 3) (b) (3, 0) (c) (0, -3) (d) (-3, 0)

7. If (0, 3) is on the graph of y = f(x), which of the following points must be on the graph of $y = \frac{1}{2}f(x)$? Multiple choice.

(a) (0, 6) (b) (0, 3) (c) $\left(0, \frac{3}{2}\right)$ (d) (3, 0)

8. The graph of function f is given below. Using the graph of f, graph each of the following functions.

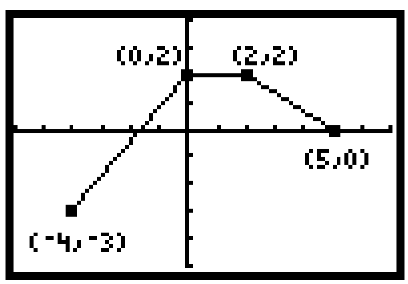
(a) P(x) = -f(x)

(b) h(x) = f(2x)

(c) F(x) = f(x) + 4

(d) G(x) = f(x – 2)

(e) H(x) = f(x + 1) – 2



Do your best! Live and learn!

Solutions:

1. (a) $y = (x - 5)^3$ (b) $y = |x| + 3$ (c) $y = -\sqrt{x + 2}$

(d) $y = 4x^2$ (e) $y = \sqrt{\frac{1}{3}x}$

2. (a) The graph of the new function is shifted left 3 units and reflected about the x-axis.

(b) The graph is vertically compressed by a factor of $\frac{1}{2}$ and shifted down 4 units.

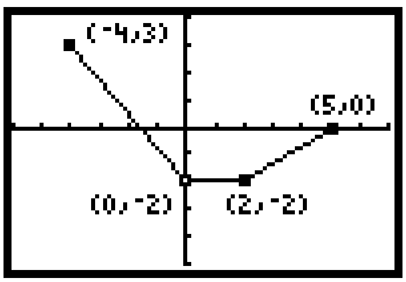
(c) The graph is vertically stretched by a factor of 2, shifted right 3 units and up 4 units.

3. (a) $y = x^3 + 6$ (b) $y = (-x)^3$ (c) $y = -x^3$

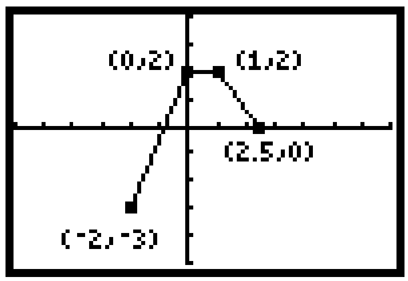
(d) $y = (x - 4)^3$ (e) $y = 1.5x^3$

4. a 5. b 6. d 7. C

8.



(a)



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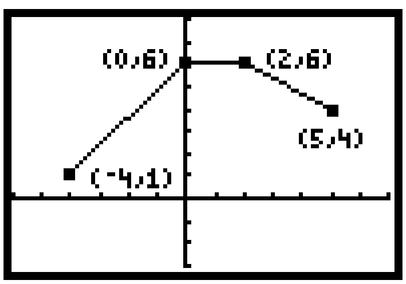
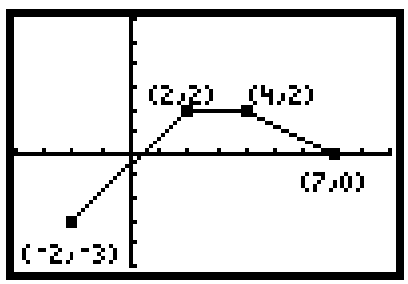
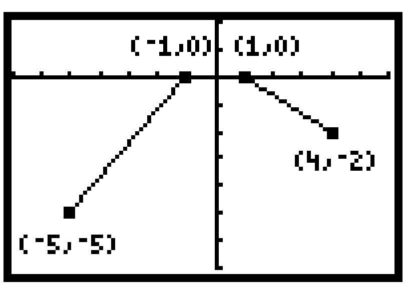


(c)



(d)



(e)