Quadratic Applications

There are many applications of quadratic functions in physics or physical science (with the study of projectile motion), in business (with profit functions), and in agriculture (with plant yields). We'll consider a few of these in depth. Many of the application questions will involve finding an x-intercept or y-intercept, the vertex, or another specific point on the graph of a parabola. These concepts were all developed earlier.

Examples/Solutions:

(1) **Projectile Motion** — The height in feet, H, of a projectile can be approximated over time in seconds, t, using the following function: $H(t) = v_0 t - 16t^2$. The constant v_0 stands for the initial velocity of the ball, rocket, cannon ball, or whatever else is being projected.

(a) Complete the table below for an initial velocity of 64 ft/sec.
The time t is in seconds, and the height H(t) is in feet.

Time (seconds)	Height (feet)
0	
0.5	
1	
1.5	
2	
2.5	
3	
3.5	
4	

Projectile Motion	$v_0 = 64 ft/s$
Time (seconds)	**Height (feet)**
0.00	0.00
0.50	28.00
1.00	48.00
1.50	60.00
2.00	64.00
2.50	60.00
3.00	48.00
3.50	28.00
4.00	0.00

$= 64 \cdot 0.5 - 16 \cdot 0.5^2 = 32 - 4 = 28$

$= 64 \cdot 2 - 16 \cdot 2^2 = 128 - 64 = 64$

$= 64 \cdot 4 - 16 \cdot 4^2 = 256 - 256 = 0$

(b) Approximate when the projectile reaches its maximum height.

We notice the clear symmetry about the point (2.00, 64) in the chart. Also, the equation of the parabola in standard form is $H(t) = -16t^2 + 64t + 0$, so a = –16, b = 64, and c = 0. Then, using the formula, we find the x-coordinate of the vertex.

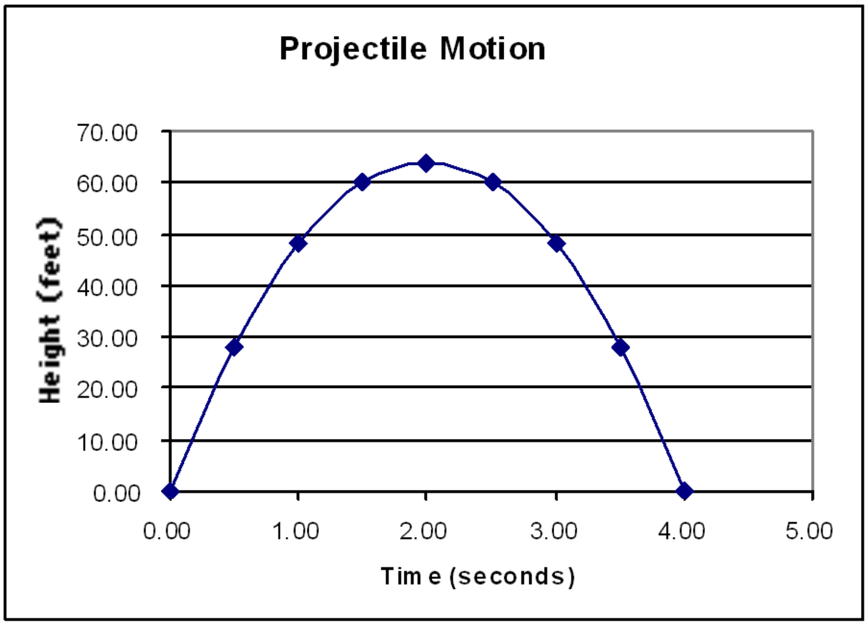
$h = -\frac{b}{2a} = -\frac{64}{2(-16)} = \frac{-64}{-32} = 2$. At 2.00 s, the projectile is at its maximum height.

(c) Approximate the projectile's maximum height.

Substitute 2 for t in the equation: $H(2) = -16(2^2) + 64(2) = 64$. The maximum height is 64 ft.

(d) Sketch the ordered pairs in the table as points, and sketch a quadratic curve that fits the data. Place the dependent variable on the vertical axis.

Height depends on the time, and a careful plotting of the points would yield a graph like the Excel graph shown below.



(e) Approximate the duration of the object's flight.

We can tell the "t-intercepts" (x-intercepts) by setting H(t) equal to 0, factoring the quadratic expression in t, and using the zero product property, as shown below:

$0 = -16t^2 + 64t$
$0 = -16t(t - 4)$
$-16t = 0$ or $t - 4 = 0$
$t = 0$ or $t = 4$

Thus, $H(0) = 0$ and $H(4) = 0$, and the duration of the object's flight is 4 seconds. Recall that if the quadratic equation doesn't factor nicely, we can always use the quadratic formula to find these x-intercepts.

In general, the y-intercept is the starting height (0 ft or ground level, in this case); the vertex involves any question involving "maximum", and the duration of flight involves x-intercepts.

(2) Another application: “The Wheat Problem”

The number of bushels of wheat an acre of land will yield depends on how many seeds per acre you plant. From previous planting statistics you find that if you plant 200,000 seeds per acre, you harvest an average of 22 bushels per acre, and if you plant 400,000 seeds per acre, you harvest an average of 40 bushels per acre. As you plant more seeds per acre, the harvest will reach a maximum, then decrease. This happens because the young plants crowd each other out and compete for food and sunlight. Assume, therefore, that the number of bushels per acre you harvest varies with the number of seeds per acre you plant according to a quadratic function.

(a) Write three ordered pairs in the form (# of hundred thousands of seeds, # of bushels). The third ordered pair is not given above, but should make good sense.

# of seeds per acre (in hundred thousands)	# of bushels per acre yield
0	0
2	22
4	40

A comment on (0, 0) — If you plant no seeds, you will not get any yield of wheat. “You reap what you sow.”

(b) Find the particular quadratic equation for this situation. Clearly show each step in the process.

The solution involves an interesting (but perhaps not commonly known) fact about parabolas. While 2 points “determine” a line, 3 points determine a parabola. If you draw 3 points on a coordinate grid (or in space, for that matter), there is exactly one parabola that contains those 3 points.

We are assuming the relationship here is quadratic. So, we start with standard form $y = ax^2 + bx + c$. The 3 points should each “satisfy” the equation. So, we substitute the coordinates for each ordered pair to find three equations with 3 unknowns: a, b, and c. Then we solve the resulting system to find a, b, and c.

Using (0, 0), we have $0 = a \cdot 0^2 + b \cdot 0 + c$, which yields the fact that c = 0.

Using (2, 22) with c = 0, we have $22 = a \cdot 2^2 + b \cdot 2 + 0$, which becomes 4a + 2b = 22.

Using (4, 40) with c = 0, we have $40 = a \cdot 4^2 + b \cdot 4 + 0$, or 16a + 4b = 40.

These two equations form a linear system that can be solved by elimination.

$$\begin{array}{ll} 16a + 4b = 40 & \qquad 16a + 4b = 40 \\ -2(4a + 2b = 22) & \qquad \underline{-8a - 4b = -44} \\ & \qquad 8a \qquad = -4 \qquad \text{So, } a = -\frac{1}{2}. \end{array}$$

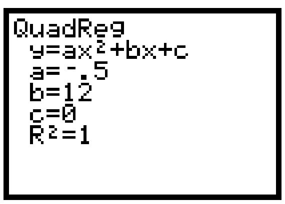
Substituting this value for a into an original equation, we have $4\left(-\frac{1}{2}\right) + 2b = 22$.

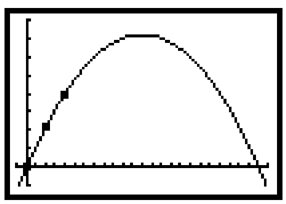
Solving for b, we have –2 + 2b = 22, then b = 12.

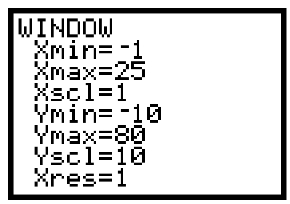
Substituting a = $-\frac{1}{2}$, b = 12, and c = 0 into the standard form for a quadratic function,

we have the equation $y = -\frac{1}{2} \cdot x^2 + 12 \cdot x + 0$ or $y = -\frac{1}{2}x^2 + 12x$.

Note: Another approach is to enter the 3 ordered pairs into 2 statistical lists and have the graphing calculator perform quadratic regression (using the QuadReg command in the STAT CALC menu). We show the quadratic regression analysis results, the scatter plot/curve, and the window inputs below.







(c) How many bushels per acre would you expect to get if you plant 800,000 seeds per acre? 1,600,000 seeds per acre?

Substituting 8 for x, $y = -\frac{1}{2}\left(8^2\right) + 12(8) = 64$ bushels/acre

Substituting 16 for x, $y = -\frac{1}{2}\left(16^2\right) + 12(16) = 64$ bushels/acre

(d) How many seeds per acre should you plant in order to get the maximum number of bushels per acre?

This is a vertex question. We may use the vertex formula:

$h = -\frac{b}{2a} = -\frac{12}{2\left(-\frac{1}{2}\right)} = -\frac{12}{-1} = 12$ The solution is, then, 1,200,000 seeds per acre.

(e) What is the maximum number of bushels per acre you can expect?

Again, this is a vertex question. Find k = f(12).

$$k = f(h) = f(12) = -\frac{1}{2}\left(12^2\right) + 12(12) = 72 \text{ bushels/acre}$$

(f) Based on your model, would it be possible to get a harvest of 60 bushels per acre? Explain your reasoning.

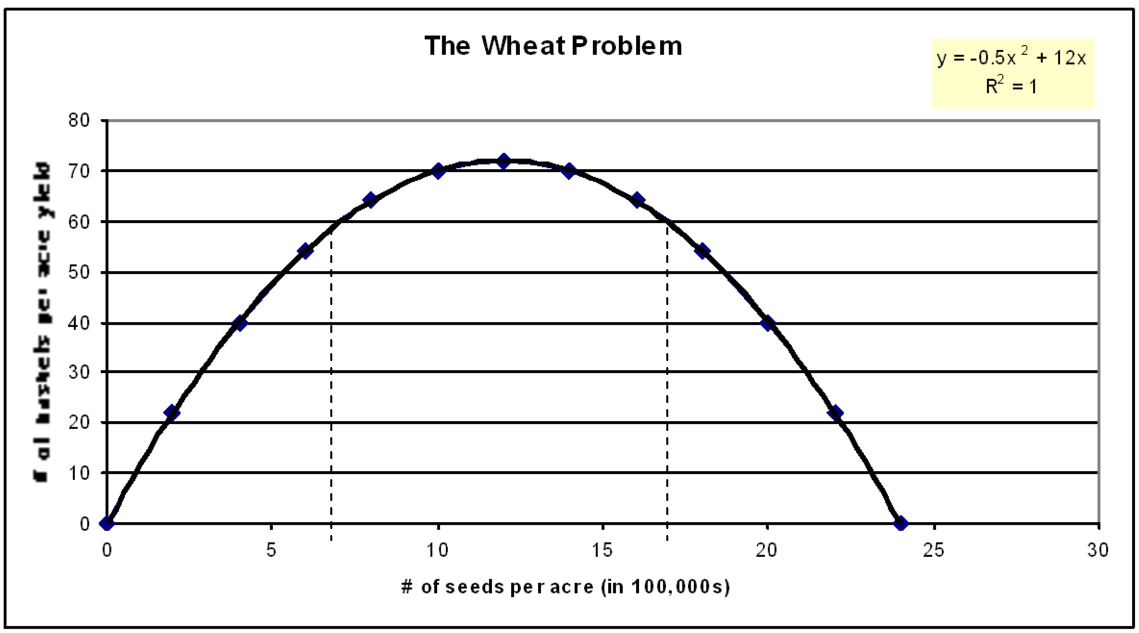
Yes. Since the vertex is (12, 72) and the orientation of the graph is downward, any y value from 0 to a maximum of 72 is practically possible. If you set y equal to 60 in the regression equation and solve for x using the quadratic formula, you will find the x values to be $12 \pm 2\sqrt{6}$. These are approximately 16.9 (or 1,690,000 seeds per acre) and 7.1 (or 710,000 seeds per acre). These are shown with dashes in the graph below.

(g) According to your model, is it possible to plant so many seeds that you harvest no wheat at all? Explain your reasoning.

Yes. Using symmetry, if (0, 0) is on the parabola and (12, 72) is the vertex, then (24, 0) is a point on the parabola. In other words, planting 2,400,000 seeds per acre would result in no wheat yield at all. Algebraically, we may set y equal to 0, factor the resulting quadratic equation, and then use the zero product property, as shown:

$$0 = -\frac{1}{2}x^2 + 12x \qquad 0 = -\frac{1}{2}x(x - 24) \qquad x = 0 \quad \text{or} \quad x = 24$$

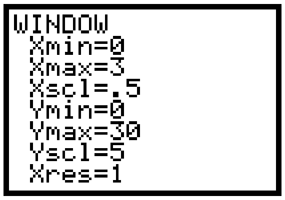
(h) Sketch the graph of this quadratic equation in a suitable domain. Clearly label the axes. Use a scale on each axis so that we can see the intercepts and the vertex.

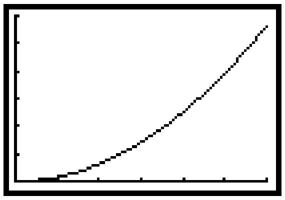


[The practical domain of this function is $0 \leq x \leq 24$ or [0, 24], and the range is $0 \leq y \leq 72$ or [0, 72].

Exercises:

1. One of the quadratic relationships we've seen before is the relationship between the radius and area of a circle: $A = \pi r^2$. The graph below shows area as a function of radius, with X representing radius and Y representing area. It is part of a parabola (over the domain r > 0). Complete the following table for circles of several different sizes. In each case, round your answers like the given measurement.





radius	Area
10 cm	
5 ft	
100 mi	
2.5 m	
	412 ft^2

2. The height, H, of a projectile can be approximated over time, t, using the following formula: $H(t) = v_0 t - 16t^2$. The time is in seconds, and the height is in feet. The constant given by v_0 is the velocity given initially to the object.

(a) Complete the table for initial velocities of 50 feet per second and 75 feet per second.

Projectile Motion	$\boxed{v_0 = 50\,ft/s}$	$\boxed{v_0 = 75\,ft/s}$
Time (seconds)	**Height (feet)**	**Height (feet)**
0.00		
0.25		
0.50		
0.75		
1.00		
1.25		
1.50		
1.75		
2.00		
2.25		
2.50		

(b) Sketch a graph of both quadratic functions on the same grid. Clearly label vertices, intercepts, and axes of symmetry for both parabolas.

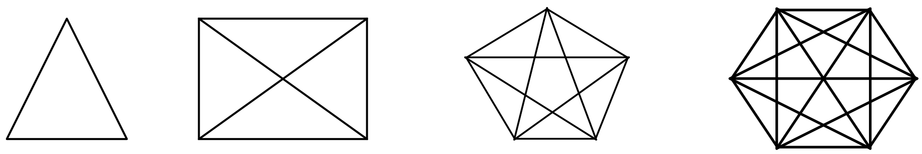
(c) Describe and compare the graphs, using language about the relative steepness, vertices, and intercepts.

3. The formula $s = 16t^2$ is used to approximate the distance s, in feet, that an object falls freely from rest in t seconds. Students drop a ball from the top of a radio tower that stands 175 ft tall. To the nearest tenth of a second, how long would it take the ball falling freely from the top to reach the ground?

4. Joanna and Lydia have a campsite. One morning, Joanna decides to bike due north, and Lydia goes due east. Lydia bikes 2 mph slower than Joanna. After 3 hours, they are 30 miles apart. Find the speed of each bicyclist to the nearest mile per hour. Mr. Pythagoras strikes again!

5. A farmer with 10,000 yards of fencing wants to enclose a rectangular field and then divide it into three plots with two fences parallel to one of the sides. What is the largest area that can be enclosed?

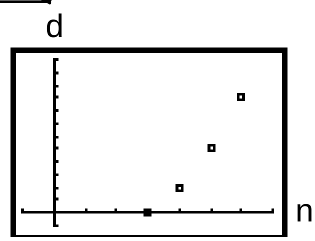
6. An interesting quadratic relationship exists between the number of sides (n) and the number of diagonals (d) of a polygon. A diagonal of a polygon connects a vertex to a non-adjacent vertex. Refer to Section 1.1 for more details on the vocabulary.



n	d
3	0
4	2
5	5
6	9

The formula relating the two variables is

$d = \frac{1}{2}n(n-3)$ or $d = \frac{1}{2}n^2 - \frac{3}{2}n$.



We include the table for the polygons shown, along with a graph of the scatter plot. This is a good example of a discrete situation; the variable n can only take on values like 3, 4, 5, 6, and so on.

For parts a, b, c, and d, find the number of diagonals, d, for the given number of sides:

(a) n = 12 (b) n = 15 (c) n = 30 (d) n = 103

*(e) Find the number of sides if the number of diagonals is 135.

7. How high on a building will a 25-foot ladder reach if its foot is 15 feet away from the base of the wall?

8. The following data came from a Calculator-Based Ranger™ (CBR) and a 15-cm rubber air-filled ball. The CBR was placed on the floor face up. The ball was thrown upward, and it landed directly on the CBR device.

Find the quadratic regression equation of best fit, and compute the correlation coefficient (rounding all coefficients to 4 decimal places).

Then use the equation to predict the maximum height of the ball to the nearest hundredth of a meter.

Time (sec)	Height (m)
0.0000	1.03754
0.1080	1.40205
0.2150	1.63806
0.3225	1.77412
0.4300	1.80392
0.5375	1.71522
0.6450	1.50942
0.7525	1.21410
0.8600	0.83173

9. A conical coffee filter is 8.4 cm tall (h = 8.4). The volume, V, of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r represents the radius of the base circle and h represents the height of the cone.

(a) Write a formula for the filter's volume in terms of its widest radius (at the top of the filter)?

(b) Complete the table of values with this volume equation. Round to the nearest tenth.

Radius, r (cm)	1	2	3	4	5
Volume, V (cm^3)					

(c) If you double the radius of the filter, by what factor does the volume increase?

(d) If the volume of the filter is 302.4 cm^3, what is the radius of the cone? Round to the nearest tenth.

10. You are driving at 60 miles per hour when you put on the brakes. Find the equation for the quadratic function that best fits the data in the table below which involves time elapsed and braking distance. Let d represent the distance in feet that your car travels in t seconds after braking. Round all coefficients to the nearest hundredth.

Time (seconds)	1	2	3	4
Distance (feet)	81	148	210	240

11. A storage box for sweaters is constructed from a square sheet of corrugated cardboard measuring x inches on a side. The volume of the box, in cubic inches, is given by the formula $V = 10(x - 20)^2$. If the box should have a volume of 1,960 cubic inches, what size cardboard square is needed?