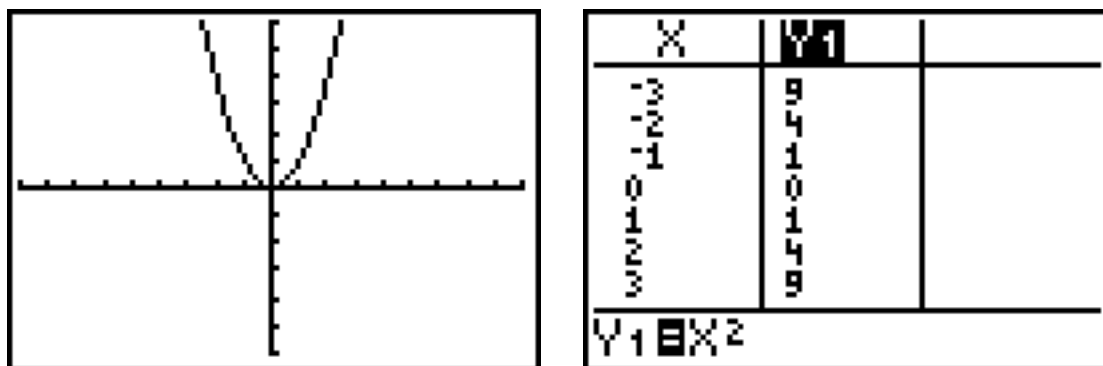# Symmetry

When we graphed $y = x$, $y = x^2$, $y = |x|$, $y = x^3$, $y = \sqrt[3]{x}$, and $y = \dfrac{1}{x}$, we mentioned some of the features of these members of the "Library of Functions", the "building blocks" for much of the study of algebraic functions. Now we go deeper into the study of symmetry, exploring three main categories or types of symmetry.

TI-83 note: We will be using a variety of window settings, but unless otherwise indicated, the scales for both the horizontal and vertical axes will be 1.

Symmetry with respect to the y-axis:

The first function we'll consider is the **squaring function** $y = x^2$ whose graph is the parabola shown below. We mentioned before that this is a simple **quadratic function**, and its main feature is its vertex, the point given by (0, 0). This graph clearly has symmetry about the y-axis (the line x = 0).


Notice the table values in pairs, like (-3, 9) and (3, 9), (-2, 4) and (2, 4), etc., and then read carefully the following generalization:

> A graph is symmetric with respect to the y-axis if, for every point (x, y) on the graph, the point (-x, y) is also on the graph.

Notice in the table that f(-3) = f(3) = 9 and that f(-2) = f(2) = 4. When f(-x) = f(x) for all real values of x in the domain of a function, the function is called an **even function**, and the graph has symmetry with respect to the y-axis.

Another function in the "library" that has symmetry with respect to the y-axis is the absolute value function, $y = |x|$, shown in the figure to the right.
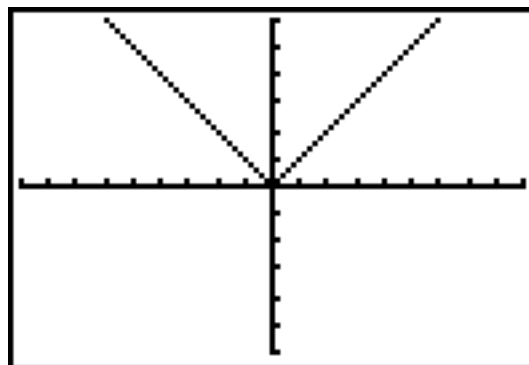
This, too, is an even function, with points such as (-3, 3) and (3, 3), (-2, 2) and (2, 2), (-1, 1) and (1, 1). The origin (0, 0) is its minimum point.

1

**Symmetry with respect to the x-axis**:

A function cannot have symmetry with respect to the x-axis, so now we turn to a few relations to illustrate this type of symmetry.

The graph of $y^2 = x$ is shown below. Compare the table for $y = x^2$ with the table below. These two relations have a special relationship; they are inverses of one another. (More on this, later!)
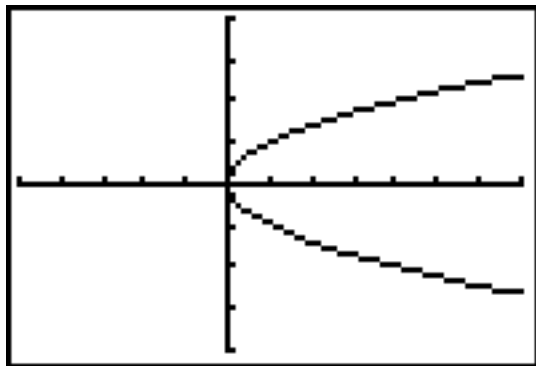
x	y
9	-3
4	-2
1	-1
0	0
1	1
4	2
9	3

Another relation that has symmetry with respect to the x-axis is $x = |y|$, shown below.
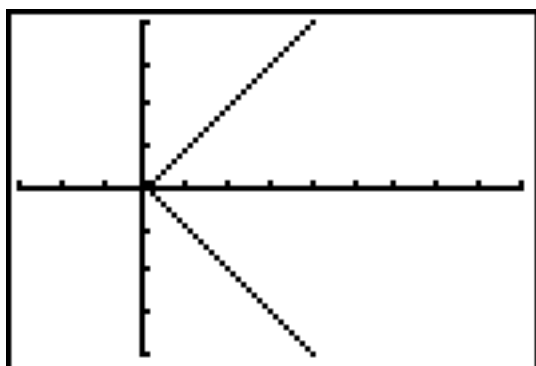
This relation includes points such as (0, 0), (1, 1), (1, -1), (2, 2), and (2, -2).

Again, notice the relationship between this graph and its "cousin", the absolute value function, $y = |x|$.

In our two examples, notice that a substitution of –y for y yields essentially the same equation: $(-y)^2 = y^2 = x$ and $x = |-y| = |y|$. Whenever this is the case, the relation has symmetry with respect to the x-axis.

> A graph is symmetric with respect to the x-axis if, for every point (x, y) on the graph, the point (x, -y) is also on the graph.

Our third and final type of symmetry to explore is the first that is not primarily reflectional, but rotational. Here we'll begin with the algebraic generalization and then move into examples.

Symmetry with respect to the origin:

> A graph is symmetric with respect to the origin if, for every point (x, y) on the graph, the point (-x, -y) is also on the graph.

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Here are a few examples.
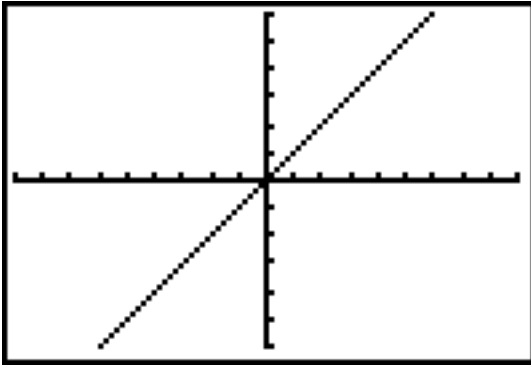
Imagine rotating the graph of y = x a full 180° about the origin. All the points in the first quadrant (points like (1, 1), (2, 2) and (5, 5)) would "land" on points in the third quadrant (on (-1, -1), (-2, -2), and (-5, -5)), and vice versa.

Notice also that both (1, 1) and (-1, -1) are points on the graph, as are (2, 2) and (-2, -2), etc.
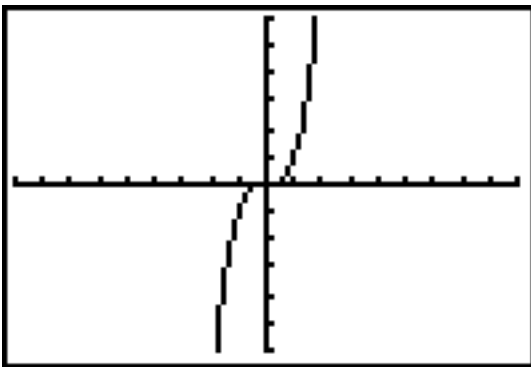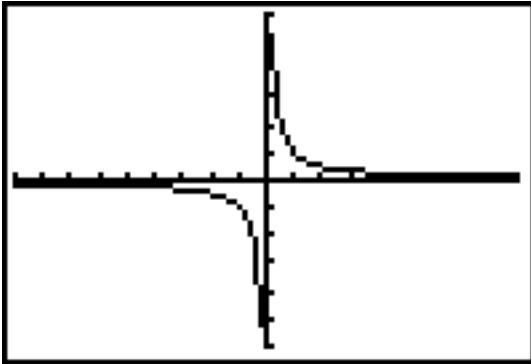

This graph is of the cubic function, $f(x) = x^3$.

Notice that f(-3) and f(3) are opposites, -27 and 27, and that a similar pattern continues through the table: f(-3) = -f(3)= -27, f(-2) = -f(2) = -8, and so on.

In general, f(-x) = -f(x).

x	y
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27

This is the rational function given by $y = \dfrac{1}{x}$. Again, notice the clear pattern in the table.

For every point (x, y) in the table, there's another point (-x, -y) also in the table.

Also notice that x cannot be 0.

x	y
-10	-0.1
-1	-1
-0.1	-10
0	---
0.1	10
1	1
10	0.1

In each of our three examples, f(-x) = -f(x). When f(-x) = -f(x) for all real values of x in the domain of a function, the function is called an **odd function** and the graph has symmetry with respect to the origin. Reflecting the graph 180° about the origin yields an identical graph.

Refer to the following chart as a summary of the odd and even function "tests":

Odd function	f(-x) = -f(x) for all x in the domain of f
Even function	f(-x) = f(x) for all x in the domain of f

It is best to simply calculate f(-x), -f(x) and compare these results with the original function, f(x).

Examples:

(1) Test each function for symmetry with respect to the y-axis, the x-axis, and the origin. Graph each function to verify your conclusions.

(a) $y = \dfrac{1}{x^2}$

Solution: Since replacing x by -x yields an equivalent function $y = \dfrac{1}{(-x)^2} = \dfrac{1}{x^2}$, the graph is symmetric with respect to the y-axis. Substituting –y for y yields $-y = \dfrac{1}{x^2}$ or $y = -\dfrac{1}{x^2}$, so this function isn't symmetric with respect to the x-axis. Similarly, replacing both –y for y and –x for x yields $-y = \dfrac{1}{(-x)^2}$ or $y = -\dfrac{1}{x^2}$, so this function also isn't symmetric with respect to the origin.

The graph below verifies symmetry with respect to the y-axis.
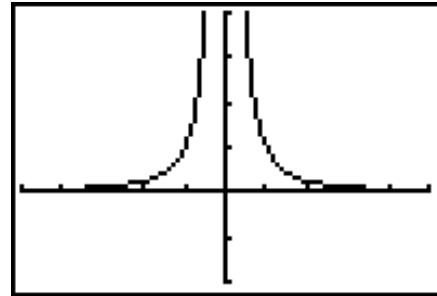
(b) $f(x) = (x - 1)^3$

Solution: Replacing x by –x yields $y = (-x - 1)^3$, which is not equivalent to the original function. The graph, therefore, won't have symmetry with respect to the y-axis.

Replacing y by –y yields $-y = (x - 1)^3$ or $y = -(x - 1)^3$, so the graph won't have symmetry with respect to the x-axis.

Replacing x by –x and y by –y yields $-y = (-x - 1)^3$, which simplifies to $y = -(-x - 1)^3$ or $y = (x + 1)^3$, so the graph won't have symmetry with respect to the origin.

By the graph, it is true that the graph has point symmetry about (1,0) which is its inflection point, but it has none of the three symmetries we're considering.

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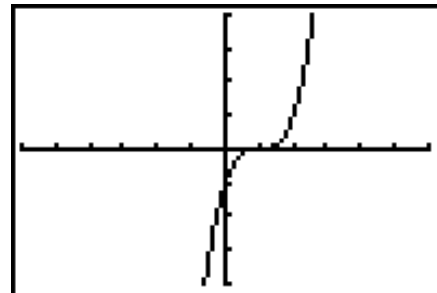
(c) $x^2 + y^2 = 4$

Solution: This circle relation has symmetry with respect to the y-axis, x-axis, and the origin. It also has reflectional symmetry over any line passing through the origin and rotational symmetry through any angle with the origin as a fixed point.

The 3 equations $(-x)^2 + y^2 = 4$, $x^2 + (-y)^2 = 4$, and $(-x)^2 + (-y)^2 = 4$ are each equivalent to $x^2 + y^2 = 4$, so we have algebraic verification for symmetry with respect to the y-axis, x-axis, and the origin, respectively.
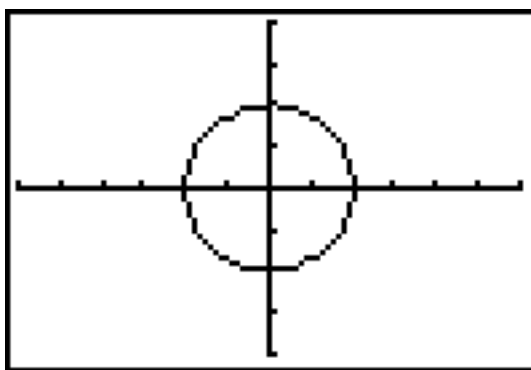
(2)   Determine algebraically whether the given function is odd, even, or neither. Graph each function to verify your conclusions.

(a) $f(x) = |3x|$

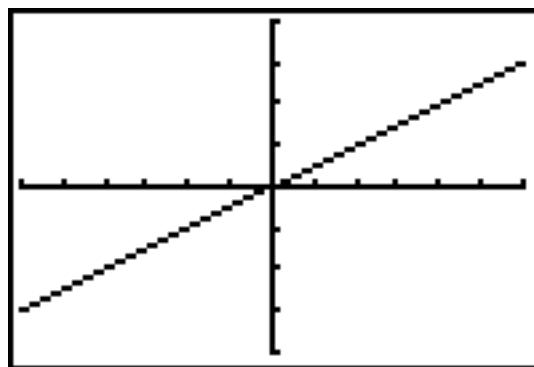Solution: $f(-x) = |3(-x)| = |3x|$
$-f(x) = -|3x|$

Since $f(x) = f(-x)$, the function is even, and its graph has symmetry with respect to the y-axis. Since $f(-x) \neq -f(x)$, the function is not odd.
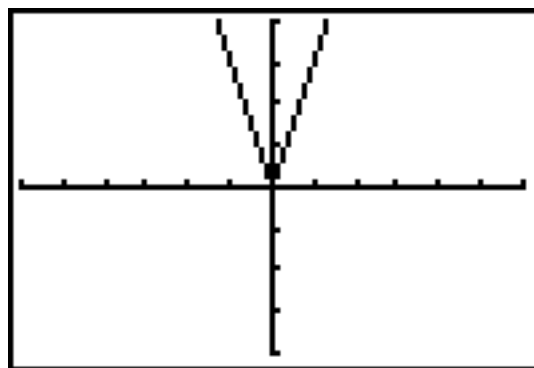
(b) $g(x) = \dfrac{1}{2}x$

Solution: $g(-x) = \dfrac{1}{2}(-x) = -\dfrac{1}{2}(x)$

$-g(x) = -\dfrac{1}{2}(x)$

Since $g(x) \neq g(-x)$, the function is not even. Since $g(-x) = -g(x)$, the function is odd, and its graph has symmetry with respect to the origin.
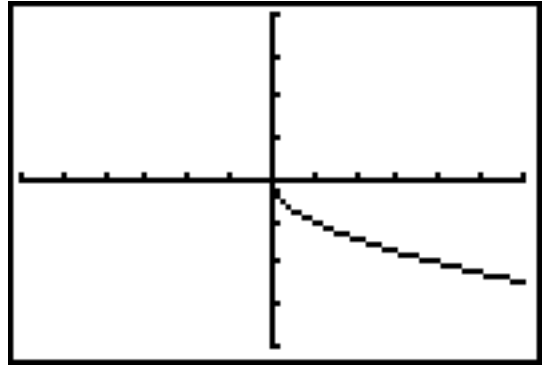

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(c) $h(x) = -\sqrt{x}$

Solution: $h(-x) = -\sqrt{-x}$
                  $-h(x) = \sqrt{x}$

Since $h(x) \neq h(-x)$, the function is not even. Since $h(-x) \neq -h(x)$, the function is not odd. The function is neither even, nor odd.
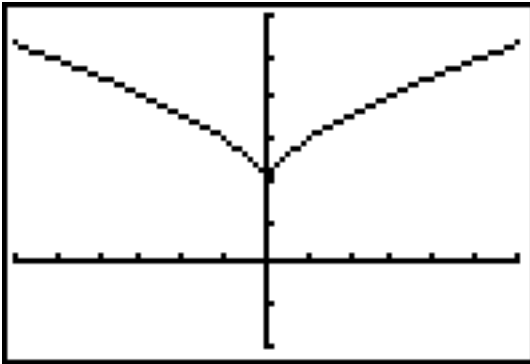
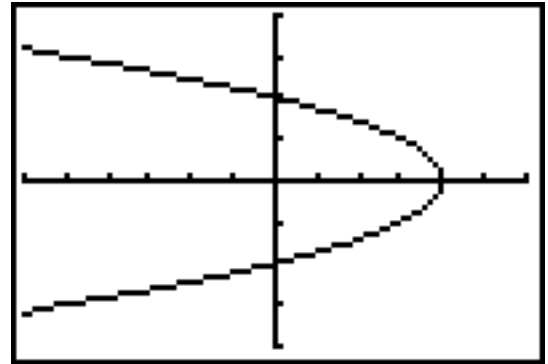These conclusions are all supported by the graph above.

Exercises:

1.   Determine visually whether the corresponding graph is symmetric with respect to the x-axis, y-axis, or origin.
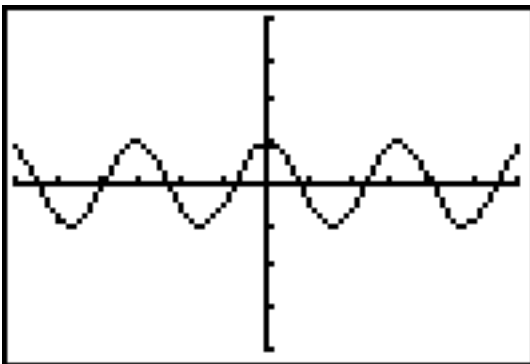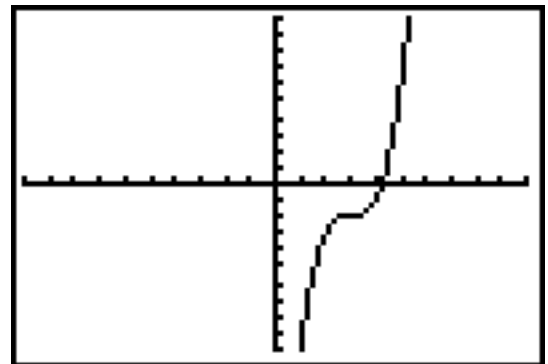
(a)

(b)

(c)

(d)

2. Graph each function to determine whether each function may have symmetry with respect to the y-axis, the x-axis, and the origin. Then verify your conclusion algebraically.

(a) $y = x^4 - 9x^2$ (b) $y = x^3 - 27$ (c) $y + 2x = 0$

(d) $y = \dfrac{x^2 - 4}{x}$ (e) $x^2 = y + 5$ (f) $x = y^2 + 5$

3. Determine algebraically whether the given function is odd, even, or neither.

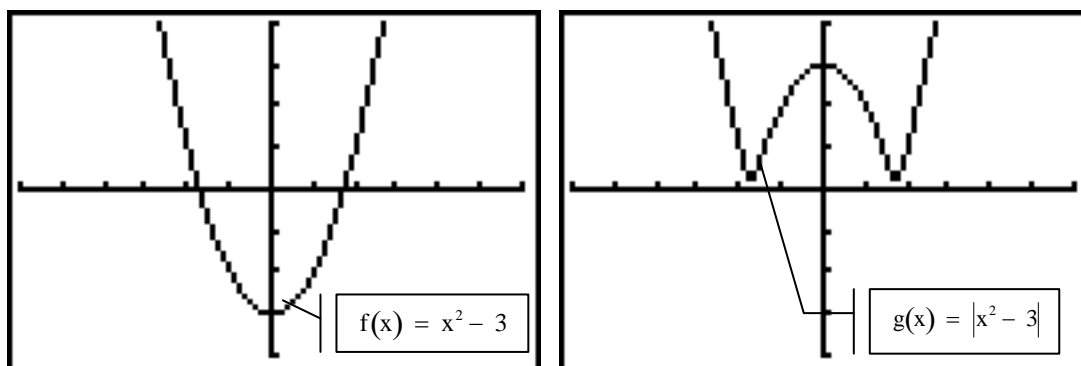(a) $f(x) = x^2 + 3$ (b) $g(x) = x + |x|$ (c) $y = \sqrt[3]{x}$

(d) $h(x) = 4x^3 - 6$ (e) $F(x) = 5$ (f) $G(x) = \sqrt[3]{2x^2 - 1}$

4. Compare the graphs of f and g.

$f(x) = x^2 - 3$

$g(x) = |x^2 - 3|$

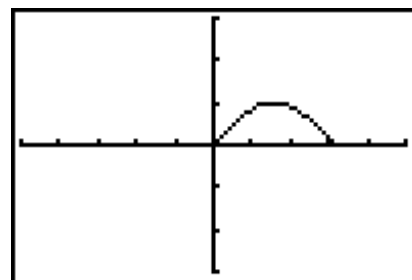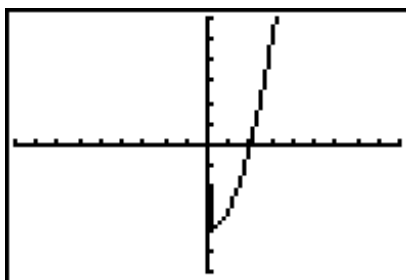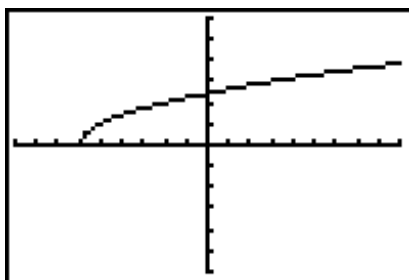(a) Describe the effect of the absolute value sign on the graph.

(b) What kind of symmetry do the two functions have in common?

5. Complete the graph below so that it has the indicated type of symmetry.

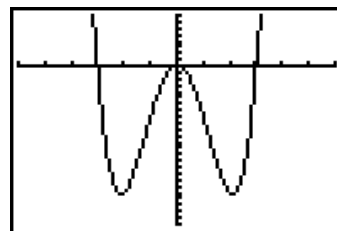(a) x-axis   (b) y-axis   (c) origin
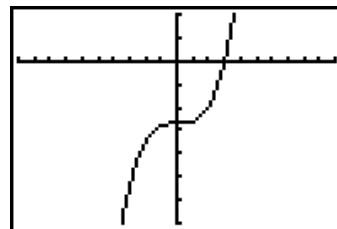
<u>Solutions</u>:

1.   (a) y-axis   (b) x-axis   (c) y-axis   (d) none

2.   (a) The quartic function, $y = x^4 - 9x^2$, has symmetry
        with respect to the x-axis, but it doesn't have either
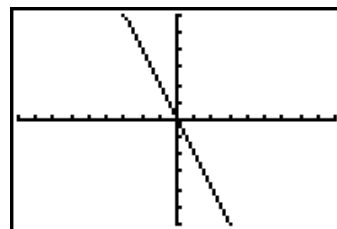   of the other two symmetries.

$$y = (-x)^4 - 9(-x)^2 = x^4 - 9x^2$$

   (b) The cubic function, $y = x^3 - 27$, has none of the
        three symmetries. It does, however, have point
        symmetry with respect to its inflection point,
        (0, -27). Note: The scale on the y-axis in the graph
        is 10.
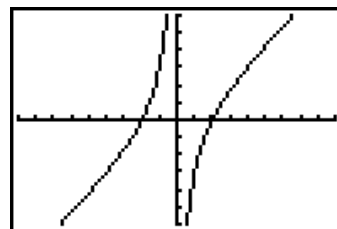
   (c) The linear function, $y + 2x = 0$, has symmetry
        with respect to the origin, but it doesn't have
        either of the other two symmetries.

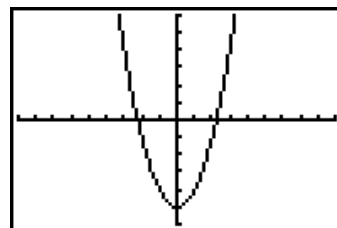$$(-y) + 2(-x) = 0 \text{ is equivalent to } y + 2x = 0.$$

   (d) The rational function, $y = \dfrac{x^2 - 4}{x}$, has symmetry
        with respect to the origin, and it has neither of
        the other two symmetries.

$$(-y) = \dfrac{(-x)^2 - 4}{(-x)} \text{ is equivalent to } y = \dfrac{x^2 - 4}{x}.$$
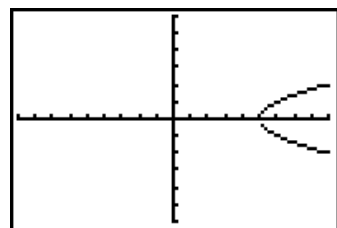
   (e) The quadratic function, $y = x^2 - 5$, has symmetry
        with respect to the y-axis and none of the other
        two symmetries.

$$y = (-x)^2 - 5 = x^2 - 5$$

   (f) The quadratic relation, $x = y^2 + 5$, has symmetry
        with respect to the x-axis and none of the other
        two symmetries.

$$x = (-y)^2 + 5 = y^2 + 5$$

3.   (a) $f(-x) = (-x)^2 + 3 = x^2 + 3$ $-f(x) = -x^2 - 3$

         Since f(-x) = f(x), the function is even; since f(-x) ≠ -f(x), the function isn't odd.

     (b) $g(-x) = (-x) + |-x| = -x + |x|$ $-g(x) = -x - |x|$

         Since g(-x) ≠ g(x) and g(-x) ≠ -g(x), the function is neither even nor odd.

     (c) Let y = f(x). $f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x}$ $-f(x) = -\sqrt[3]{x}$

         Since f(-x) ≠ f(x), the function isn't even; since f(-x) = -f(x), the function is odd.

     (d) $h(-x) = 4(-x)^3 - 6 = -4x^3 - 6$ $-h(x) = -4x^3 + 6$

         Since h(-x) ≠ h(x) and h(-x) ≠ -h(x), the function is neither even nor odd.

     (e) F(-x) = 5   -F(x) = -5

         Since F(-x) = F(x), the function is even; since F(-x) ≠ -F(x), the function isn't odd.

     (f) $G(-x) = \sqrt[3]{2(-x)^2 - 1} = \sqrt[3]{2x^2 - 1}$ $-G(x) = -\sqrt[3]{2x^2 - 1}$

         Since G(-x) = G(x), the function is even; since G(-x) ≠ -G(x), the function isn't odd.

4.   (a) The portion of the graph of f which is below the x-axis seems to be reflected upward over the x-axis, resulting in the graph of g.

     (b) Both are even functions and have symmetry with respect to the y-axis.
         Both f(-x) = f(x) and g(-x)= g(x).

5.   (a)   (b)

     (c)