MATH 1001 (Quantitative Skills and Reasoning)
Gordon State College
Supplement to Chapter 9

Quadratic Functions and Modeling

In this unit, we will study quadratic functions and the relationships for which they provide suitable models. An important application of such functions is to describe the trajectory, or path, of an object near the surface of the earth when the only force acting on the object is gravitational attraction. What happens when you toss a ball straight up into the air? What about an outfielder on a baseball team throwing a ball into the infield? If air resistance and outside forces are negligible, what is the mathematical model for the relationship between time and height of the ball?

Definition: ***Quadratic Functions***

A **quadratic function** is one of the form

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real numbers with $a \neq 0$.

The graph of a quadratic function is called a ***parabola*** and its shape resembles that of the graph in each of the following two examples.

Example 1

Figure 1 shows the graph of the quadratic function

$$y = f(x) = x^2 - 4x + 1.$$

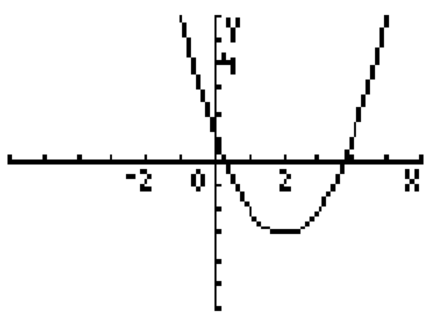

figure 1

Observe that there is a lowest point $V(2, -3)$ on the graph in figure 1. The point V is called the ***vertex*** of the parabola.

Example 2

Figure 2 shows the graph of the quadratic function

$$y = g(x) = -2x^2 + 4x + 3.$$

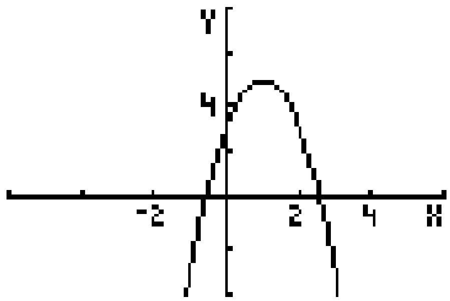

figure 2

Again, observe that there is a highest point $V(1, 5)$ on the graph in figure 2. This point V is also the ***vertex*** of the parabola.

By **completing the square**,
the quadratic function (example 1) $f(x) = x^2 - 4x + 1$ can be written as

$$f(x) = (x-2)^2 - 3;$$

the quadratic function (example 2) $g(x) = -2x^2 + 4x + 3$ can be written as

$$g(x) = -2\,(x-1)^2 + 5\,.$$

Note that the parabola in example 1 opens upward, with vertex $V(2, -3)$ and a vertical axis of symmetry $x = 2$. The parabola in example 2 opens downward with vertex $V(1, 5)$ and a vertical axis of symmetry $x = 1$.

Standard Form of a Quadratic Function Whose Graph (a Parabola) has a Vertical Axis

The graph of $$\begin{aligned} f(x) &= a x^2 + bx + c \quad , \quad a \neq 0 \\ &= a(x-h)^2 + k \end{aligned}$$

is a parabola that has vertex $V(h, k)$ and a vertical axis.

The parabola opens upward if $a > 0$ or downward if $a < 0$.

If $a > 0$, the vertex $V(h, k)$ is the lowest point on the parabola, and the function f has a minimum value $f(h) = k$.

If $a < 0$, the vertex $V(h, k)$ is the highest point on the parabola, and the function f has a maximum value $f(h) = k$.

Quadratic Models and Equations

Quadratic Population Models:

$$P(t) = P_0 + bt + at^2 .$$

Here, we denote the independent variable by t (time) instead of x, and the constant c by P_0 because substitution of $t = 0$ yields $P(0) = P_0$.
We refer to P_0 as the initial population.

Example 3

Suppose that the future population of Stockton City t years after January 1, 2000 is described (in thousands) by the quadratic model

$$P(t) = 110 + 4t + 0.07t^2 .$$

(a) What is the population of Stockton City on January 1, 2007?
(b) In what month of what calendar year will the population of Stockton City reach 180 thousand?

Solution

(a) We only need to substitute $t = 7$ in the population function $P(t)$ and calculate

$$P(7) = 110 + 4(7) + 0.07(7)^2 = 141.43 \text{ thousand.}$$

(b) We need to find the value of t when $P(t) = 180$. That is we need to solve the equation,

$$110 + 4t + 0.07t^2 = 180 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{(i)}$$

Equation (i) is an example of a **quadratic equation** which we can solve in a **variety of ways**.

First, by **graphing** both sides of the equation (figure 3):

$$Y_1 = 110 + 4t + 0.07t^2$$

$$Y_2 = 180$$

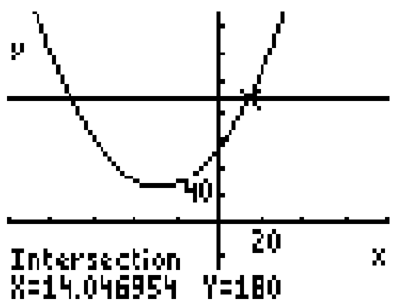

figure 3

The line $Y_2 = 180$ and the parabola $Y_1 = 110 + 4x + 0.07x^2$ are shown in the calculator window $-100 \le x \le 80$, $-100 \le y \le 300$.

To solve equation (i), we find the x-coordinate of the intersection point in the first quadrant. The negative solution of the intersection point in the second quadrant would be in the past. Figure 3 indicates that we have already used the intersection-finding feature to locate the point (14.047, 180).

Hence, t = 14.047 years
= 14 years + (0.047 x 12) months
= 14 years + 0.56 month.

Thus Stockton City should reach a population of 180 thousand 14 years, 0.56 month after January 1, 2000.

Hence, sometime during January 2014.

Alternatively, by using the **Quadratic Formula:**

The quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$
has solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

To use the quadratic formula, we first write equation (i) in the form

$$0.07t^2 + 4t - 70 = 0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{(ii)}$$

Here, $a = 0.07$, $b = 4$, $c = -70$, giving

$$t = \frac{-4 \pm \sqrt{4^2 - 4(0.07)(-70)}}{2(0.07)}$$
$$= \frac{-4 \pm \sqrt{35.6}}{0.14} = 14.046954 \quad or \quad -71.189811$$

The negative solution would be in the past. So, we only accept the positive solution, $t =$ 14.047 years = 14 years, 0.56 month.

The Position Function (Model) of a Particle Moving Vertically:

If an object is projected straight upward at time $t = 0$ from a point y_0 feet above ground, with an initial velocity v_0 ft/sec, then its height above ground after t seconds is given by

$$y(t) = -16t^2 + v_0 t + y_0 .$$

Example 4

A projectile is fired vertically upward from a height of 600 feet above the ground, with an initial velocity of 803 ft/sec.

(a) Write a quadratic model for its height $y(t)$ in feet above the ground after t seconds.
(b) During what time interval will the projectile be more than 5000 feet above the ground?
(c) How long will the projectile be in flight?
(d) How long does it take to reach its maximum height?
(e) What is the maximum height that the object reaches?

Solution

(a) Given $y_0 = 600$ ft and $v_0 = 803$ ft/sec,

$$y(t) = -16t^2 + 803t + 600.$$

(b) We need to find the values of t for which the height $y(t) > 5000$ feet.
That is

$$-16t^2 + 803t + 600 > 5000 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{(i)}$$

We solve the inequality (i) by graphing (**figure 4**) both the parabola $Y_1 = -16t^2 + 803t + 600$ and the line $Y_2 = 5000$, in the calculator window $-20 \le x \le 70$ and $-2000 \le y \le 12000$.

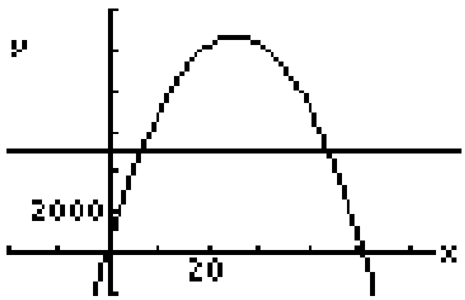

figure 4

The parabola in figure 4 shows the height of the projectile at time t and the horizontal line is the graph of a height of 5000 feet.

The projectile is more than 5000 feet above the ground when the graph of the parabola is above the horizontal line.

Using the intersection-finding feature of the calculator, we find the approximate intersection points to be $t = 6.3$ sec and $t = 43.9$ sec.

Hence, the projectile is above 5000 feet when

$$6.3 \text{ sec} < t < 43.9 \text{ sec.}$$

(c) The projectile will be in flight until its height $h(t) = 0$. This corresponds to the x-intercept (not the origin) in figure 4.

Using the root or zero feature of the calculator, we obtain

$$t = 50.9 \text{ sec.}$$

(d) To find how long the object takes to reach the zenith, we will to find the first-coordinate of the vertex of the parabola given by $y(t) = -16t^2 + 803t + 600$. Completing the square is very difficult in this case so we will rely on the graphing calculator to find when the maximum occurs. Below is the procedure for this.

1. Enter the quadratic function in **Y1**. (Use "*x*" as the variable.)
2. Graph it and set the appropriate viewing window.
3. Press **2nd**, **CALC** and select **4:maximum**.
4. Use the left arrow to go to a place on the left of the maximum or minimum. Press **ENTER**.
5. Use the right arrow to go to a place on the right of the maximum or minimum. Press **ENTER**.
6. Then go near the maximum or minimum. Press **ENTER**.

After this procedure your display should look like this.

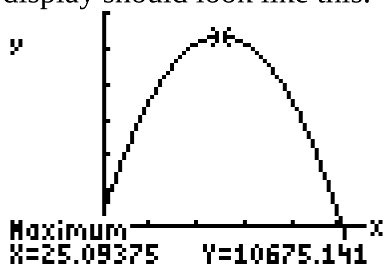

figure 5

The *x*-coordinate is the time the object takes to reach its maximum height. Hence, the object reaches its maximum height at approximately 25.1 seconds.

(e) Using the same process as in part (d), the *y*-coordinate gives the maximum height of the object. Thus, the object reaches a maximum height of approximately 10,675.1 feet.

EXERCISES

1. The population (in thousands) for Alpha City, t years after January 1, 2004 is modeled by the quadratic function $P(t) = 0.3t^2 + 6t + 80$. In what month of what year does Alpha City's population reach twice its initial (1/1/2004) population?

2. The population (in thousands) for Beta City, t years after January 1, 2005 is modeled by the quadratic function $P(t) = 0.7t^2 + 12t + 200$. How long will it take Beta City's population to reach 350 thousand?

3. The population (in thousands) for Gamma City, t years after January 1, 2002 is modeled by the quadratic function $P(t) = 1.5t^2 + 21t + 300$. How long will it take Gamma City's population to reach 500 thousand?

4. The population (in thousands) for Delta City, t years after January 1, 2003 is modeled by the quadratic function $P(t) = 0.5t^2 + 7t + 90$. In what month of what year does Delta City's population reach twice its initial (1/1/2003) population?

5. The population (in thousands) for Omega City, t years after January 1, 2002 is modeled by the quadratic function $P(t) = 0.25t^2 + 5t + 100$. In what month of what year does Omega City's population reach 200 thousand?

6. A ball is thrown straight up, from ground zero, with an initial velocity of 48 feet per second. Find the maximum height attained by the ball and the time it takes for the ball to return to ground zero.

7. From the top of a 48 foot tall building, a ball is thrown straight up with an initial velocity of 32 feet per second. Find the maximum height attained by the ball and the time it takes for the ball to hit the ground.

8. A ball is thrown straight up from the top of a 160 foot tall building with an initial velocity of 48 feet per second. The ball soon falls to the ground at the base of the building. How long does the ball remain in the air?

9. A ball is dropped from the top of a 960 foot tall building. How long does it take the ball to hit the ground?

10. Joshua drops a rock into a well in which the water surface is 300 feet below ground level. How long does it take the rock to hit the water surface?

For #11-15, find the quadratic model $P(t) = P_0 + bt + at^2$ (with $t = 0$ for the earliest year given in the data) that best fits the population census data. In each case, calculate the average error of this optimal model, and use the model to predict the population in the year 2007.

11. Iowa City, IA

t (years)	1970	1980	1990	2000
P (people)	46,850	50,508	59,735	62,220

12. Arizona

t (years)	2000	2001	2002	2003	2004
P (thousands)	5,131	5,320	5,473	5,581	5,744

13. Florida

t (years)	1970	1980	1990	2000	2004
P (thousands)	6,789	9,746	12,938	15,982	17,346

14. Georgia

t (years)	1970	1980	1990	2000	2004
P (thousands)	4,590	5,463	6,478	8,186	8,825

15. U.S.

t (years)	1970	1980	1990	2000	2004
P (millions)	203	227	249	281	294

16. The following data came from a Calculator-Based Ranger™ (CBR) and a 15-cm rubber air-filled ball. The CBR was placed on the floor face up. The ball was thrown upward, and eventually it landed directly on the CBR device.

Find the quadratic regression equation of best fit, and compute the correlation coefficient (rounding all coefficients to 4 decimal places).

Then use the equation to predict the maximum height of the ball to the nearest hundredth of a meter.

Time (sec)	Height (m)
0.0000	1.03754
0.1080	1.40205
0.2150	1.63806
0.3225	1.77412
0.4300	1.80392
0.5375	1.71522
0.6450	1.50942
0.7525	1.21410
0.8600	0.83173

17. The following data involve the greatest single vertical drop in feet and the maximum speed in miles per hour for various roller coasters.

Roller Coaster	Location	x (ft)	y (mph)
Scream Machine	Atlanta, GA	89	57
Texas Giant	Arlington, TX	137	65
Hercules	Allentown, PA	151	65
American Eagle	Gurnee, IL	147	66
Alpengeist	Williamsburg, VA	170	67
Magnum XL-200	Cedar Point, OH	195	72
Son of Beast	Kings Mill, OH	214	78
Silver Star	Rust, GERMANY	220	79
The Desperado	Primm, NV	225	80
Phantom's Revenge	Kennywood, PA	228	82
Millennium Force	Cedar Point, OH	300	93
Superman The Escape	Valencia, CA	328	100

Find the quadratic regression equation of best fit, and compute the correlation coefficient (rounding all coefficients to 4 decimal places).

Then use the equation to predict the maximum speed of a roller coaster if its greatest single vertical drop is 400 feet.

Answers to Exercises

1. February, 2013
2. 8 years, 143 days
3. 6 years, 183 days
4. February, 2011
5. May, 2014
6. 36 feet, 3 sec
7. 64 feet, 3 sec
8. 5 sec
9. 7.75 sec
10. 4.33 sec
11. $P(t) = 46{,}234 + 641t - 2.9t^2$; $1377\ \mathit{people}$; $66{,}000\ \mathit{people}$
12. $P(t) = 5{,}139 + 176t - 6.9t^2$; $14\,\mathit{thousand}$; $6{,}030\,\mathit{thousand}$
13. $P(t) = 6{,}777 + 298t + 0.37t^2$; $44\,\mathit{thousand}$; $18{,}300\,\mathit{thous\,and}$
14. $P(t) = 4{,}610 + 59t + 1.93t^2$; $59\,\mathit{thousand}$; $9{,}400\,\mathit{thousand}$.
15. $P(t) = 204 + 2t + 0.02t^2$; $1.4\ \text{million}$; $300\ \text{million}$.
16. $H(t) = -4.6764t^2 + 3.7583t + 1.0450$; r ≈ 0.9997; 1.80 m
17. $y = -0.0001394x^2 + 0.1238x + 44.3557$; r ≈ 0.9943; 116 mph

Note: Answers to all questions are approximate values.

References

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   R. David Gustafson & Peter D. Frisk. Thomson (Brooks/Cole), 2005.

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