Graphing Quadratic Functions

In our consideration of polynomial functions, we first studied linear functions. Now we will consider polynomial functions of order or degree 2 (i.e., the highest power of x in the equation is 2). These are called **quadratic functions**, and their graph is called a **parabola**. The parabola is a commonly used shape for a variety of applications, including reflecting mirrors in car headlights and cable supports for suspension bridges. Other common applications involve agriculture, business, and the study of projectile motion (useful in many sports).

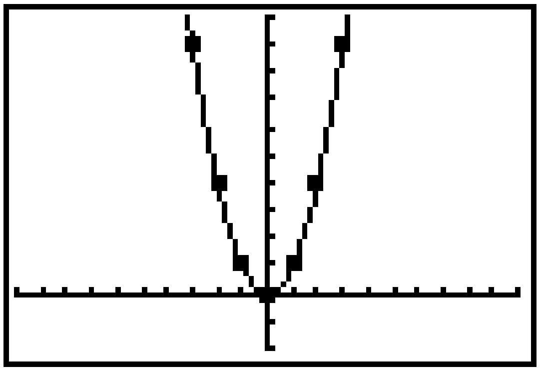
Quadratic functions have two equation forms we will consider. Both forms are given in the chart below, along with examples of equations in each form.

Forms	Examples
General form: $y = ax^2 + bx + c, \quad a \neq 0$	$y = x^2$, $y = 3x^2 - x - 6$, $y = -\frac{1}{2}x^2 + 2x - 8$
Vertex (or standard) form: $y = a(x - h)^2 + k$	$y = x^2$, $y = 2(x - 3)^2 + 4$, $y = -\frac{1}{3}(x + 5)^2 - 8$

We'll start with the most important quadratic function ("a building block"): $y = x^2$.

Squaring -3, -2, -1, 0, 1, 2, and 3 yields the y-values shown in the chart. Plot the points and draw a continuous curve that fits them.

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



A few features to notice about the graph:

- The curve is symmetric about the y-axis (the line x = 0 is called the line of symmetry or the **axis of symmetry**). You see the reflectional symmetry both on the graph and in the chart.

- The lowest or minimum point is called the **vertex**. Here, the vertex is the origin (0, 0).

- This is an example of an **even function** with f(-x) = f(x) for all x in the domain of f. For instance, f(-2) = f(2) = 4 and f(-5) = f(5) = 25.

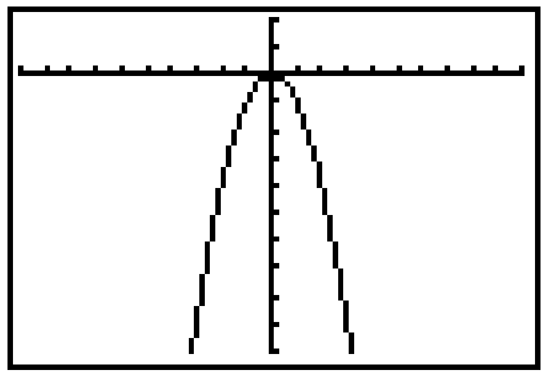
- This curve has **decreasing** y-values for x values less than 0 and **increasing** y-values for x values greater than 0.

Now, we'll consider the vertex form of the equation of quadratic functions, $y = a(x - h)^2 + k$. Step by step, we will see how the values of a, h, and k in the vertex form affect the graph of the parabola.

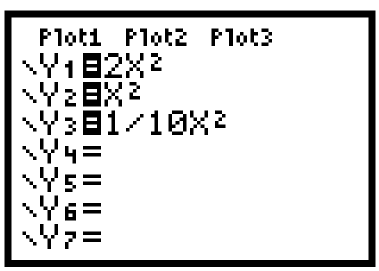
Starting with the value of a, we'll first consider several graphs of quadratic functions of the form $y = ax^2$.

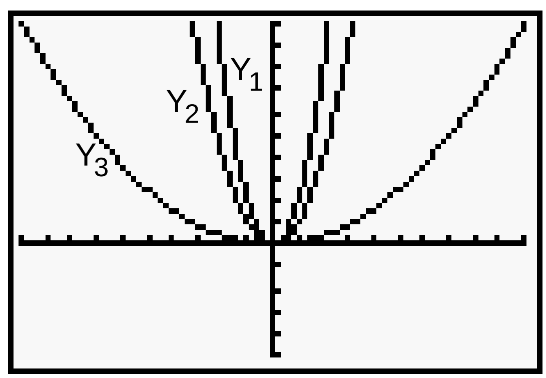
To graph the parabola given by $y = -x^2$ (where a = -1), we need to create a table of values. Notice that only the x-value gets squared, and the negative sign applies to the result. For instance, for x = -3, we have $y = -(-3)^2 = -9$.

x	y
-3	-9
-2	-4
-1	-1
0	0
1	-1
2	-4
3	-9



Next, we consider the graphs of $y = 2x^2$, $y = x^2$, and $y = \frac{1}{10}x^2$.





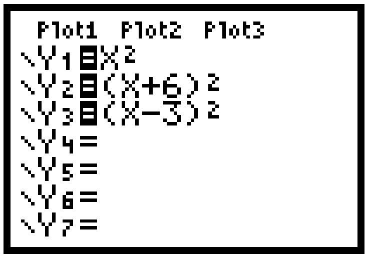
By these examples, the constant a evidently affects 2 features of the graph:

(1) **orientation** — When a > 0, the graph is oriented upwards (and the vertex is the lowest or **minimum point** on the curve); when a < 0, the graph is oriented downwards (and the vertex is the highest or **maximum point** on the curve). [This feature is also called **concavity**. The graph is concave upward if a > 0 and concave downward if a < 0.]

(2) **shape** — While all the graphs are parabolas, as the value of a increases (from 1 to 2) the graph tightens or contracts (a vertical stretch); as the value of a decreases (from 1 to 0.1, the graph widens or expands (a vertical compression).

Notice that the vertex remains (0, 0) in all of the graphs above.

Our next step — Let's see the effect of h on the graph of the parabola. The following functions all have the general form $y = a(x - h)^2$ with a = 1.



Compare the charts for Y_1, Y_2, and Y_3.

x	y1
-2	4
-1	2
0	0
1	2
2	4

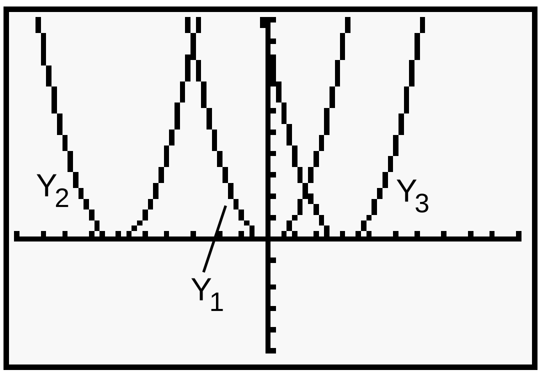
x	y2
-8	4
-7	2
-6	0
-5	2
-4	4

x	y3
1	4
2	2
3	0
4	2
5	4

With Y_1, the vertex is (0, 0), and h = 0 because the equation is equivalent to $y = (x - 0)^2$.

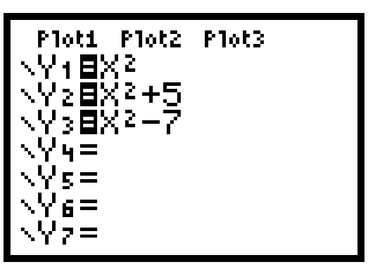
With Y_2, the vertex is (-6, 0), and h = -6 because the equation $y = (x + 6)^2$ is equivalent to $y = (x - (-6))^2$. Every point on the graph of $y = x^2$ is shifted left 6 units.

With Y_3, the vertex is (3, 0), and h = 3 because the equation is $y = (x - 3)^2$. Every point on the graph of $y = x^2$ is shifted right 3 units.



Notice that h affects a **horizontal translation**. The entire curve is shifted right if h > 0 and shifted left if h < 0.

Our final step — Let's see the effect of k on the graph of the parabola. The vertex or standard form will now be $y = a(x - h)^2 + k$. We'll let a = 1 and h = 0, and let k vary.



Compare the charts for Y_1, Y_2, and Y_3.

x	y1
-2	4
-1	2
0	0
1	2
2	4

x	y2
-2	9
-1	7
0	5
1	7
2	9

x	y3
-2	-3
-1	-6
0	-7
1	-6
2	-3

With Y_1, the vertex is (0, 0), and k = 0 because the equation is equivalent to $y = (x - 0)^2 + 0$. [This is the "building block" for parabolas.]

With Y_2, the vertex is (0, 5), and k = 5 because the equation is equivalent to $y = (x - 0)^2 + 5$. Every point on the graph of $y = x^2$ is shifted up 5 units.

With Y_3, the vertex is (0, -7), and k = -7 because the equation is equivalent to $y = (x - 0)^2 - 7$. Every point on the graph of $y = x^2$ is shifted down 7 units.

Notice that k affects a **vertical translation**. The entire curve is shifted up if $k > 0$ and shifted down if $k < 0$.

Here's a summary up to this point.

If the equation of a quadratic function is in vertex form, $y = a(x - h)^2 + k$, the value of a in the formula tells us the orientation (up if $a > 0$ and down if $a < 0$), and a also affects the shape of the parabola (a vertical stretch or compression). The variables h and k together tell us where on the grid the parabola is located; in general, the vertex of the parabola is given by the ordered pair (h, k). The axis or line of symmetry is the vertical line through the vertex given by the equation x = h.

Now we'll revisit the general form of a quadratic function and see its relationship to vertex or standard form with a few examples.

<u>Examples</u>:

(1) The equation of a parabola in vertex form is $y = f(x) = (x - 3)^2 + 4$. Here, the vertex is clearly the point (3, 4); the axis of symmetry is the line given by the equation x = 3; and since a = 1, the parabola opens upward and has the exact same shape as $y = x^2$.

Expanding the equation yields $y = (x - 3)(x - 3) + 4$.
Then, using "F O I L", we have $y = x^2 - 3x - 3x + 9 + 4$.
Combining like terms, we have $y = x^2 - 6x + 13$.

This equation is now in general form, $y = f(x) = ax^2 + bx + c$, with a = 1, b = -6, and c = 13. While the starting and ending equations are clearly in different forms (vertex form and standard form), the graphs are identical.

Notice that the value a in standard form is the same as the one in the vertex form; regardless of which form the equation is written, if a = 1, our parabola opens upward and has the exact same shape as $y = x^2$.

The value c is significant because f(0) = c always. The c value then is the **y-intercept**. The parabola in our example has a y-intercept of 13 or (0, 13).

While vertex form clearly gives the vertex of the parabola, standard form doesn't give us the ordered pair for the vertex quite as readily. We may, however, find the vertex using a couple of formulas:

> When the equation of a parabola is in general form, $y = ax^2 + bx + c$, the vertex is given by the ordered pair (h, k), where h can be found from the values of a and b using the formulas $h = -\frac{b}{2a}$ and k = f(h).

These clever formulas may be proven either with the famous quadratic formula and the symmetry of the graph or by completing the square to change the form of the equation.

Examples:

(2) Change the quadratic equation from $y = 2x^2 - 3x + 5$ to vertex form. Then give the vertex of the parabola.

Solution: This equation is in general form, so we know that a = 2 and b = –3, and we have $h = -\frac{b}{2a} = -\frac{-3}{2(2)} = \frac{3}{4}$ or 0.75, and, substituting 0.75 into the equation for x, we have $k = f(h) = f\left(\frac{3}{4}\right) = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 5 = 3\frac{7}{8}$.

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The vertex, then, is, $\left(\frac{3}{4}, 3\frac{7}{8}\right)$ or $(0.75, 3.875)$, and, substituting a = 2, $h = \frac{3}{4}$, and $k = 3\frac{7}{8}$ into the vertex form for a quadratic function, the equation of the parabola is $y = 2\left(x - \frac{3}{4}\right)^2 + 3\frac{7}{8}$.

By the way, since c = 5, the y-intercept of the graph is (0, 5).

We'll now focus on x-intercepts and y-intercepts.

Finding a **y-intercept** for the graph of a quadratic function written in vertex form will be simply a matter of setting x equal to 0 and solving for y. In general form, the y-intercept is the value c or (0, c).

For instance, the y-intercept of for the graph of $y = 2(x + 3)^2 - 4$ is found by setting x equal to 0 and solving for y, as shown: $y = 2(0 + 3)^2 - 4 = 14$. The y-intercept is, then, 14 or (0, 14).

For the parabola given by $y = -5x^2 + 3x - 2$, the y-intercept is –2 or (0, –2).

Finding **x-intercepts** involves setting y equal to 0 and solving for x. A function has a **zero** at the value n if f(n) = 0. If the zero is a real number, the terms "zero" and "x-intercept" are interchangeable. The most important zeros for a function are the x-intercepts of its graph. There are several basic principles that are helpful in solving quadratic equations, all summarized in the chart below.

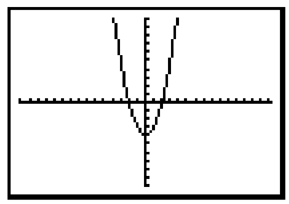
Zero Product Property	If $a \cdot b = 0$, then $a = 0$ or $b = 0$.
When is it useful?	The equation is in standard form and factors nicely.
Square Root Property	If $(x - h)^2 = r$, then $x - h = \pm\sqrt{r}$ and $x = h \pm \sqrt{r}$.
When is it useful?	The equation is in vertex form.
Quadratic Formula	If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
When is it useful?	The equation is in standard form.

Examples:

(3) Find the x-intercepts or zeros of the function given by $y = f(x) = x^2 - 4$.

Solution: Setting y = 0, we have $0 = x^2 - 4$. Add 4 to both sides, then use the square root property.

$4 = x^2$
$\pm\sqrt{4} = x$
$\pm 2 = x$



The x-intercepts are (2, 0) and (–2, 0).

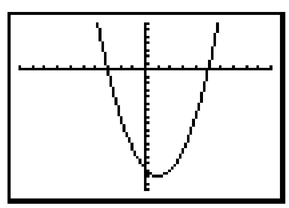
[Note: For this problem, another effective method would be to factor the difference of squares as (x + 2)(x – 2) and then use the zero product property.]

(4) Find the x-intercepts (zeros) of the function $y = f(x) = x^2 - 2x - 15$.

Solution: Setting y = 0, we have $0 = x^2 - 2x - 15$.

Factoring the right side, we can then use the zero product property.

$0 = (x - 5)(x + 3)$
$x - 5 = 0$ or $x + 3 = 0$
$x = 5$ or $x = -3$



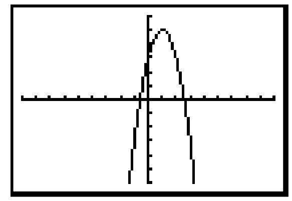
The x-intercepts are (5, 0) and (–3, 0).

(5) Find the x-intercepts or zeros of the function $y = -2x^2 + 4x + 3$.

Solution: Setting y = 0, we have $0 = -2x^2 + 4x + 3$.
This is in standard form. Notice that a = –2, b = 4, and c = 3.
Using the quadratic formula, we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(-2)(3)}}{2(-2)} = \frac{-4 \pm \sqrt{40}}{-4} = \frac{-4 \pm 2\sqrt{10}}{-4}$$

$$x = \frac{2 \pm \sqrt{10}}{2} \text{ or } 1 \pm \frac{\sqrt{10}}{2}$$



The x-intercepts are
$\left(1 + \frac{\sqrt{10}}{2}, 0\right)$ and $\left(1 - \frac{\sqrt{10}}{2}, 0\right)$,
or, approximately, (2.58, 0) and (–0.58, 0).

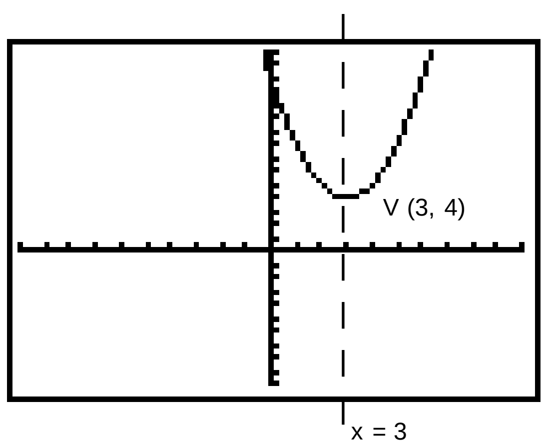
Note: The quadratic formula always works for any equation in general form. We could use it in Example 3, setting a = 1, b = 0, and c = –4, and we could use it in Example 4, setting a = 1, b = –2, and c = –15.

Now let’s “put it all together” as we graph quadratic functions with a few examples. We will include any x- and y-intercepts, the vertex, and axis of symmetry whenever they can be found. These are the 4 major features of the graph of a parabola.

(6) Graph the parabola given by $y = (x - 3)^2 + 4$. [This is Example 1, revisited and expanded.]

First, find the vertex. The equation is in vertex form, so h = 3, k = 4, and the vertex is given by (h, k) or (3, 4). Make a chart, using symmetry around the vertex. Then plot the points and draw a smooth, continuous curve. The axis of symmetry runs vertically through the vertex and has equation x = h or x = 3. Since a = 1, the parabola opens upward.

x	y
0	13
1	8
2	5
3	4
4	5
5	8



We see in the chart (and graph) that this parabola intercepts the y-axis at c = 13 or (0, 13). We can also find this by substituting 0 for x and solving for y: $y = (0-3)^2 + 4 = 13$.

To search for x-intercepts, we may set y equal to 0 and try the square root property.

$$0 = (x-3)^2 + 4$$
$$-4 = (x-3)^2$$
$$\pm\sqrt{-4} = x - 3$$
$$3 \pm \sqrt{-1}\cdot\sqrt{4} = x$$
$$3 \pm 2i = x$$

Here, we use the fact that $\sqrt{-1} = i$. The value i is called the "imaginary unit".

Notice that we are forced to find the square root of a negative number. The 2 solutions are imaginary zeros of the function (and will check in the equation), but as we can see by the graph, there are no real zeros and no x-intercepts for this parabola.

(7) Graph the parabola given by $y = -(x+2)^2 + 5$.

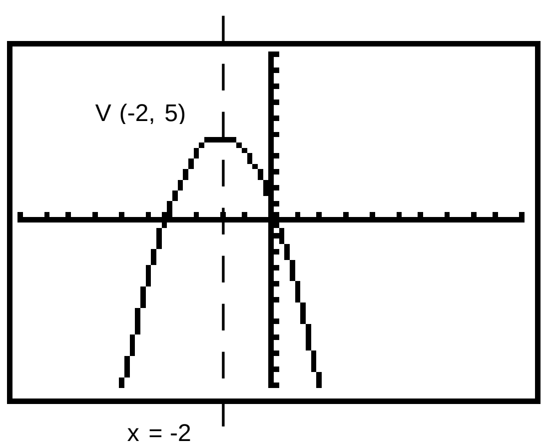
First, find the vertex. The equation is in vertex form; since h = –2 and k = 5, the vertex is (–2, 5). Make a chart, using symmetry around the vertex. Then plot the points and draw a smooth, continuous curve.

The axis of symmetry is x = –2. Since a = –1, we expect the parabola to open downward. Setting x equal to 0, we find that the parabola has a y-intercept of 1 or (0, 1): $y = -(0+2)^2 + 5 = (-4) + 5 = 1$. Setting y equal to 0 and again using the square root property, we find both x-intercepts to be irrational.

$$0 = -(x+2)^2 + 5$$
$$(x+2)^2 = 5$$
$$x + 2 = \pm\sqrt{5}$$
$$x = -2 \pm \sqrt{5}$$

A table of values and graph for the quadratic function are shown below.
Notice the axis of symmetry, the vertex, the y-intercept, and the x-intercepts. The x-intercepts are approximately –4.24 and 0.24.

x	y
–5	–4
–4	1
–3	4
–2	5
–1	4
0	1
1	–4



(8) Graph the parabola given by $y = 2x^2 - 4x + 5$.

First, find the vertex. We use the “the opposite of b over 2a” formula and the quadratic equation below, first substituting a = 2 and b = –4.

$$h = -\frac{b}{2a} = -\frac{-4}{2(2)} = \frac{4}{4} = 1 \qquad k = f(h) = f(1) = 2(1)^2 - 4(1) + 5 = 3$$

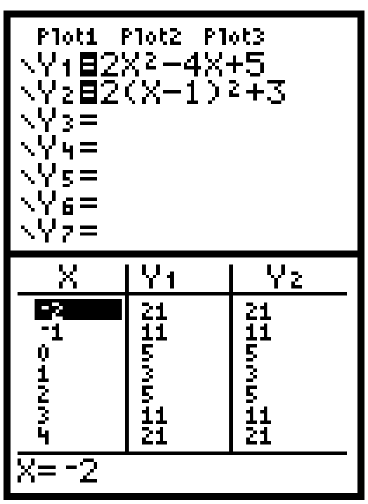
The vertex, then, is (1, 3), and the axis of symmetry is the line given by x = 1. Knowing that a = 2, h = 1, and k = 3, we may write the vertex form for this quadratic function: $y = 2(x-1)^2 + 3$.

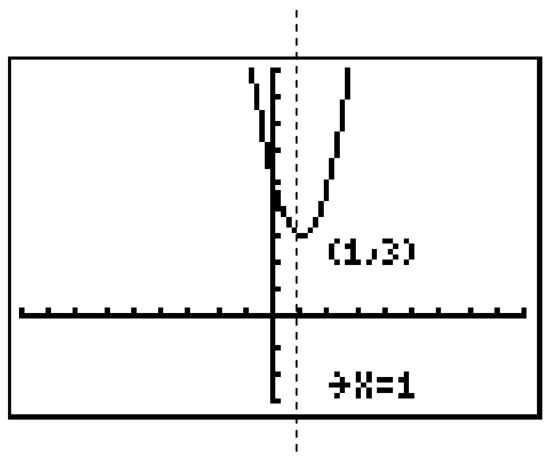
The y-intercept is the c value in the original equation, which is 5 or (0, 5). To find any x-intercepts, we set y equal to 0, and solve the equation $0 = 2x^2 - 4x + 5$ for x. Using the quadratic formula with a = 2, b = –4, and c = 5, we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(5)}}{2(2)} = \frac{4 \pm \sqrt{-24}}{4} = \ldots = \frac{2 \pm i\sqrt{6}}{2}$$

We therefore find no real zeros, so there are no x-intercepts.

We enter both equation forms in the graphing calculator, show the identical tables, and the complete graph below.



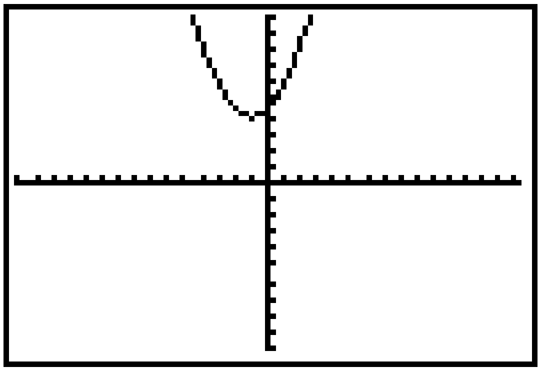


To summarize, we have two different forms for the very same parabola. The general form, $y = ax^2 + bx + c$, tells us the y-intercept directly, and the vertex can be found with a few, quick calculations. The axis of symmetry comes along with the vertex, and x-intercepts are easy to find if the expression factors and a bit harder if we must use the quadratic formula. Vertex form, $y = a(x-h)^2 + k$, directly tells us the vertex and axis of symmetry. We must work with the square root property to get the x-intercept(s), and we may use a quick “plug and chug” to get the y-intercept.

Exercises:

1. In each case, the graph of a parabola is given. Choose the most reasonable quadratic equation in vertex form.

(a)

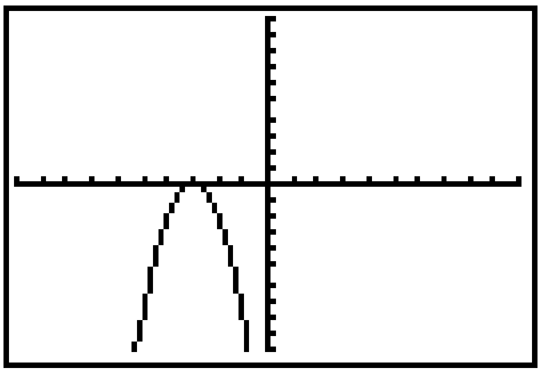


$y = \frac{1}{2}(x-1)^2 - 4$

$y = \frac{1}{2}(x+1)^2 + 4$

$y = \frac{1}{2}(x-1)^2 + 4$

(b)

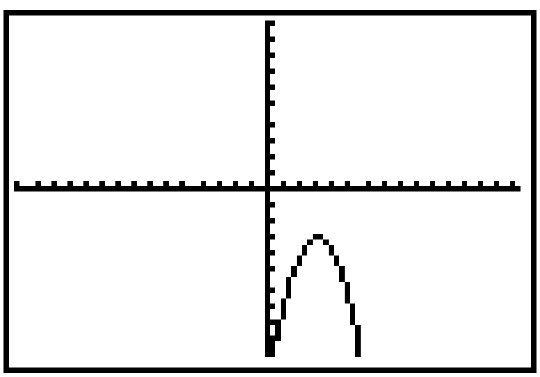


$y = -2(x+3)^2$

$y = 2(x+3)^2$

$y = -2(x-3)^2$

(c)



$y = -(x+3)^2 - 3$

$y = -(x-3)^2 - 3$

$y = (x-3)^2 - 3$

2. Find the vertex and the axis of symmetry for each quadratic function. Also decide whether the vertex is the lowest point on the curve (minimum) or the highest point on the curve (maximum).

(a) $y = 3(x+1)^2 - 4$ (b) $y = 2(x-3)^2$ (c) $y = -2(x-3)^2 - 4$

(d) $y = -x^2 + 3x - 5$ (e) $y = \frac{1}{3}x^2 - 2$

3. Find the vertex and the axis of symmetry for each quadratic function. Also decide whether the vertex is the maximum or minimum point on the graph of the parabola.

(a) $y = -(x + 3)^2 - 3$ (b) $y = 4x^2 + 6x - 3$ (c) $y = x^2 - 4x - 3$

(d) $y = \frac{1}{3}(x + 3)^2 + 3$ (e) $y = (x + 1)^2$

4. Find the x- and y-intercepts of each function, whenever they exist. Give exact solution(s).

(a) $y = 3(x + 1)^2 - 4$ (b) $y = 2(x - 3)^2$ (c) $y = -2(x - 3)^2 - 4$

(d) $y = -x^2 + 3x - 5$ (e) $y = \frac{1}{3}x^2 - 2$

5. Find the x- and y-intercepts of each function, whenever they exist. Give exact solution(s).

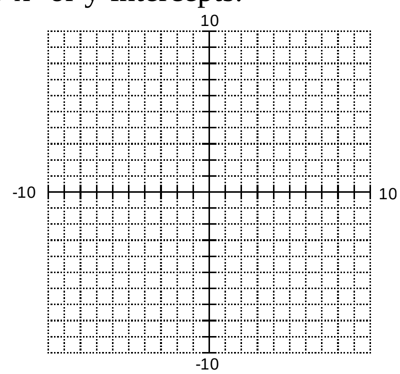
(a) $y = -(x + 3)^2 - 3$ (b) $y = 4x^2 + 6x - 3$ (c) $y = x^2 - 4x - 3$

(d) $y = \frac{1}{3}(x + 3)^2 + 3$ (e) $y = (x + 1)^2$

6. Graph the following quadratic functions on graph paper like the grid shown below. Clearly label the vertex, the axis of symmetry, and any x- or y-intercepts.

(a) $y = 3(x + 1)^2 - 4$ (b) $y = -x^2 + 3x - 5$

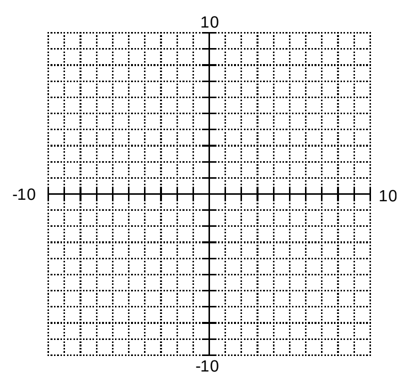
(c) $y = \frac{1}{3}x^2 - 2$ (d) $y = x^2 - 6x + 8$



7. Graph the following quadratic functions on graph paper like the grid shown below. Clearly label the vertex, the axis of symmetry, and any x- or y-intercepts.

(a) $y = -2(x - 3)^2 - 4$ (b) $y = x^2 - 4x - 3$

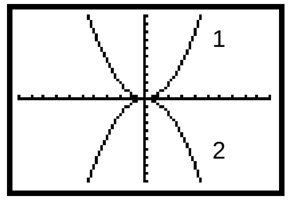
(c) y = Ğ$(x + 3)^2 + 4$ (d) $y = x^2 - 9$



8. Match the graph with its equation:

(a) $y = \frac{1}{2}x^2$ ______

(b) $y = -\frac{1}{2}x^2$ ______



9. Describe the relationship between graph of the given function and the graph of $y = x^2$ using terms such as horizontal or vertical shifting (left, right, up and/or down), vertical stretching or shrinking, and/or reflection over the x-axis.

(a) $y = \frac{1}{4}x^2$ (b) $y = -(x+2)^2 - 3$ (c) $y = (x-3)^2 + 5$

(d) $y = x^2 + 6x + 2$ (e) $y = -2(x-3)^2$

10. On one set of coordinate axes, graph the given family of parabolas. Describe how these graphs change as the value of c or b changes.

(a) $y = x^2 + 2x + c$ for $c = -3$, $c = 0$, and $c = 1$

(b) $y = x^2 + bx + 1$ for $b = -2$, $b = 0$, and $b = 2$

11. Use your graphing calculator to graph each set of quadratic functions. Describe the pattern or relationship you notice.

(a) $y = x^2 + 1$, $y = 2x^2 + 2$, and $y = 3x^2 + 3$

*(b) $y = x^2 - 9$ and $y = |x^2 - 9|$

*12. The points (0, 2), (2, 3), and (–3, 5) are plotted on the grid. Sketch a parabola that contains all 3 points. Then use the general form ($y = ax^2 + bx + c$), substitute each ordered pair for x and y in order to have 3 equations with 3 unknowns, then solve the linear system to find the coefficients (a, b, and c) of the quadratic functions which contains all 3 points.