

Graphing Exponential Functions

Another commonly used type of non-linear function is the exponential function. We focus in this section on exponential graphs and solving equations. Another section is loaded with practical applications involving exponential functions.

What distinguishes an exponential function from a polynomial function is the placement of the variable x . With exponential functions, the variable x is in the exponent. The general form for an **exponential function** is $y = b^x$, where b is a real number greater than 0 and not equal to 1. The “not equal to 1” constraint is due to the fact that $y = 1^x$ is equivalent to the horizontal line given by the equation $y = 1$ (a linear or constant function). The “greater than 0” constraint is due to the impossibility of finding a real square root (or any even root) of a negative number. For example, if we let $b = -5$ and try an x value like $1/2$, we have $y = (-5)^{1/2} = \sqrt{-5}$, which is undefined in the real number system.

Here are a few power rules, along with an example for each rule, as reminders. [Note: We assume that the values for a and b are real numbers, and the values for m and n are integers.]

| Rule | Example | Rule | Example |
|---------------------------------|----------------------------|--|--|
| $a^m \cdot a^n = a^{m+n}$ | $5^2 \cdot 5^4 = 5^6$ | $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ | $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$ |
| $\frac{a^m}{a^n} = a^{m-n}$ | $\frac{x^{12}}{x^4} = x^8$ | $a^{-n} = \frac{1}{a^n}$ | $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$ |
| $a^0 = 1$ | $10^0 = 1$ | $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ | $\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^4 = \frac{3^4}{2^4} = \frac{81}{16}$ |
| $(a^m)^n = a^{m \cdot n}$ | $(3^2)^4 = 3^8$ | | |
| $(a \cdot b)^n = a^n \cdot b^n$ | $(-3x)^2 = 9x^2$ | $\frac{1}{a^{-n}} = a^n$ | $\frac{1}{x^{-2}} = x^2$ |

We'll consider $y = 2^x$ first with a table of values, along with a few problems showing the process.

| x | y |
|----|------------------|
| -4 | $1/16$ or 0.0625 |
| -3 | $1/8$ or 0.125 |
| -2 | $1/4$ or 0.25 |
| -1 | $1/2$ or 0.5 |
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16} = 0.0625$$

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4} = 0.25$$

$$2^0 = 1$$

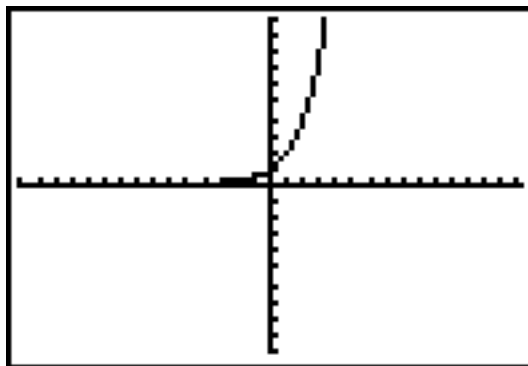
$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

$$2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$$

We can also raise the base 2 to fractional powers and decimal powers; for example,

$$2^{1/2} = \sqrt{2} \approx 1.4142 \qquad 2^{5/3} = \sqrt[3]{2^5} = \sqrt[3]{32} = \sqrt[3]{8} \cdot \sqrt[3]{4} = 2\sqrt[3]{4} \approx 3.1748 \qquad 2^{-5.43} \approx 0.0232$$

This work gives us 3 more points — $(0.5, 1.4142)$, $(\sqrt{1.6}, 3.1748)$, and $(-5.43, 0.0232)$ — for the graph. Plotting these points and a continuous curve through them yields the following curve.



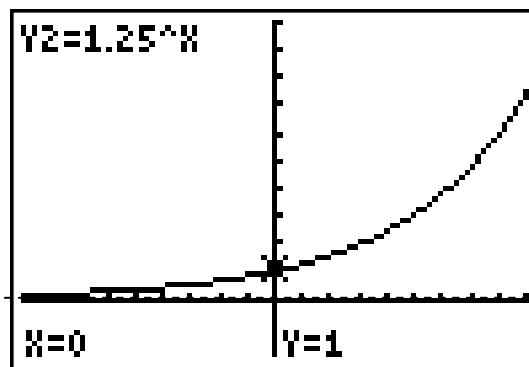
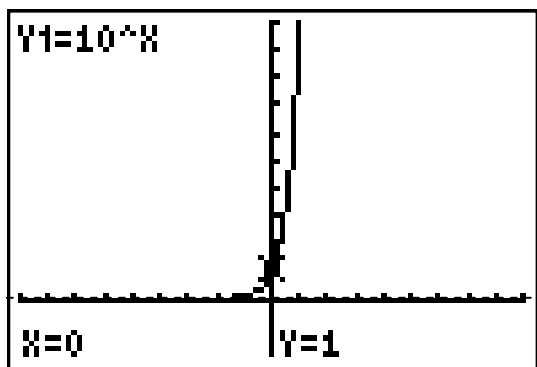
There are 3 main features to notice about the graph of $y = 2^x$:

- This is an **increasing function** over its entire domain (of all real numbers). Notice that b is 2. Exponential functions increase whenever $b > 1$. The graph is a smooth and continuous curve, increasing dramatically over its domain.

For fun, raise 2 to various powers and see how large the results are (then see what will “overflow” the calculator).

- The curve has **y-intercept** $(0, 1)$, since $2^0 = 1$. In general, any number raised to the 0 power is 1.
- The graph approaches the x-axis (the line $y = 0$) as x gets smaller and smaller. The line given by $y = 0$ is the **horizontal asymptote** for this exponential curve.

Continuing the theme of graphing exponential functions of the form $y = b^x$, with $b > 0$, consider the graphs of $y = 10^x$ and $y = 1.25^x$ below.



In general, with exponential functions of the form $y = b^x$, when $b > 1$, the function is an increasing function over its domain of all real numbers; the curve has y-intercept $(0, 1)$; and the line $y = 0$ is its horizontal asymptote.

Now we'll consider the general case when $0 < b < 1$. More specifically, we'll first let $b = \frac{1}{2}$. The

exponential function $y = \left(\frac{1}{2}\right)^x$ contains the following points, and we show the process for a few examples, as well. This equation is equivalent to $y = 2^{-x}$.

| x | y |
|----|--------------------------|
| -4 | 16 |
| -3 | 8 |
| -2 | 4 |
| -1 | 2 |
| 0 | 1 |
| 1 | $\frac{1}{2}$ or 0.5 |
| 2 | $\frac{1}{4}$ or 0.25 |
| 3 | $\frac{1}{8}$ or 0.125 |
| 4 | $\frac{1}{16}$ or 0.0625 |

$$\left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^2 = 4$$

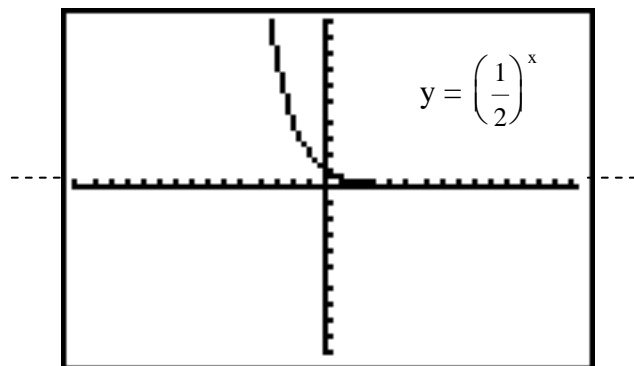
$$\left(\frac{1}{2}\right)^1 = \frac{1}{2} = 0.5$$

$$\left(\frac{1}{2}\right)^4 = \frac{1^4}{2^4} = \frac{1}{16} = 0.0625$$

Plotting the points in our table and a continuous curve through these points gives a graph like the one shown to the right. Also compare both the tables and

the graphs for $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$.

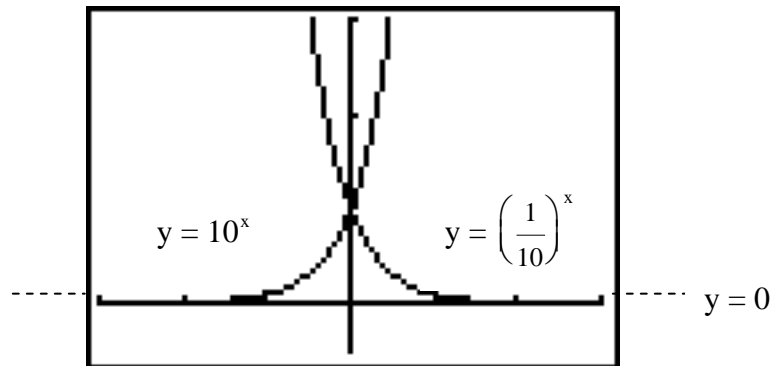
These functions are mirror reflections of one another over the y-axis.



Two of the main features of this curve are the same as the ones we discussed for $y = 2^x$.

- This is a **decreasing function** over its entire domain (of all real numbers). [This is true whenever $0 < b < 1$.]
- The curve has **y-intercept** $(0, 1)$, since $\frac{1}{2}$ raised to the 0 power is 1.
- The graph approaches the x-axis (the line $y = 0$) as x increases. The line given by $y = 0$ is a **horizontal asymptote** for the curve.

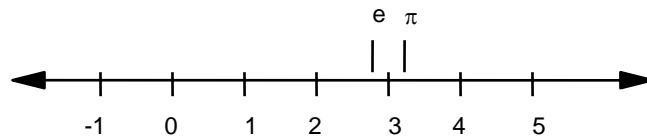
Just as $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ are mirror reflections of one another over the y-axis, the same is true for $y = 10^x$ and $y = \left(\frac{1}{10}\right)^x$, as we see in the graph below.



In general, with exponential functions of the form $y = b^x$, when $0 < b < 1$, the function is a decreasing function over its domain of all real numbers; the curve has y-intercept $(0, 1)$; and the line $y = 0$ is its horizontal asymptote.

The number e:

One of the most interesting and useful numbers in many branches of science and mathematics is the number e . It pops up in several natural settings and several business applications. We'll see it in population growth and compound interest applications in the next section. The number e is an irrational number, very much like the number π (which is approximately 3.14159). They are fairly close to one another on the real number line.



The number e is defined as the number that the expression $\left(1 + \frac{1}{m}\right)^m$ approaches as m gets larger and larger, approaching infinity. Using calculus notation, as $m \rightarrow \infty$, $\left(1 + \frac{1}{m}\right)^m \rightarrow e$.

Consider the charts below. The first foreshadows compound interest applications.

| | | | | | |
|----------------------------------|----------------------------------|----------------------------------|--------------------------------------|--------------------------------------|--|
| $\left(1 + \frac{1}{1}\right)^1$ | $\left(1 + \frac{1}{2}\right)^2$ | $\left(1 + \frac{1}{4}\right)^4$ | $\left(1 + \frac{1}{12}\right)^{12}$ | $\left(1 + \frac{1}{52}\right)^{52}$ | $\left(1 + \frac{1}{360}\right)^{360}$ |
| 2 | 2.25 | 2.44141 | 2.61304 | 2.692597 | 2.7145161 |

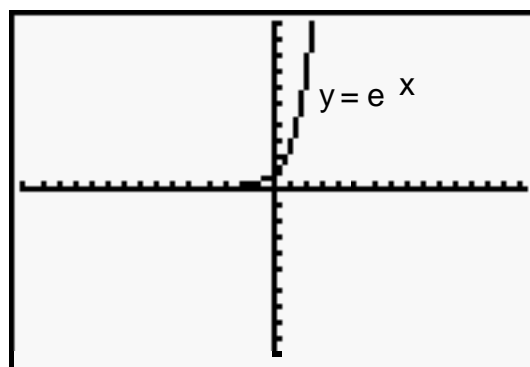
The second chart has m continue to approach infinity.

| | | |
|--|--|--|
| $\left(1 + \frac{1}{1,000}\right)^{1,000}$ | $\left(1 + \frac{1}{1,000,000}\right)^{1,000,000}$ | $\left(1 + \frac{1}{1,000,000,000}\right)^{1,000,000,000}$ |
| 2.7169239 | 2.7182805 | 2.718281827 |

Notice the trend in these answers; as m increases, the value of $\left(1 + \frac{1}{m}\right)^m$ approaches a certain number. This number, named e, is approximately 2.7182818284 It is an irrational number, neither repeating nor terminating as a decimal.

The graph of $y = e^x$ would then be “sandwiched” between the graphs of $y = 2^x$ and $y = 3^x$.

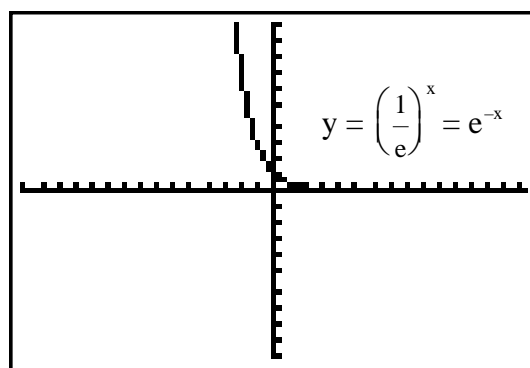
Since the base is greater than 1, this is an increasing function over its domain of all real numbers. The curve has y-intercept (0, 1), because any number (including the number e) raised to the 0 power is 1. Like the rest, it has the x-axis ($y = 0$) as its horizontal asymptote.



Similarly, the graph of $y = \left(\frac{1}{e}\right)^x = e^{-x}$ would be “sandwiched” between the graphs of $y = \left(\frac{1}{2}\right)^x$

and $y = \left(\frac{1}{3}\right)^x$.

Here, since the base is less than 1, this is a decreasing function over its domain of all real numbers. The curve also has y-intercept (0, 1). Like the rest, it has the x-axis ($y = 0$) as its horizontal asymptote.

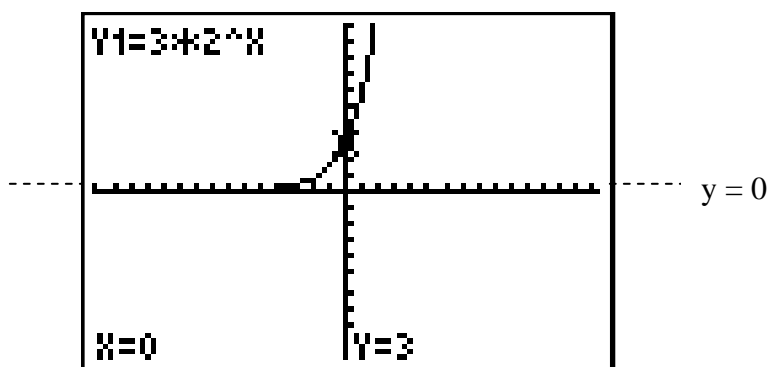


Let’s consider the exponential function given by $y = 3 \cdot 2^x$. The form $y = a \cdot b^x$ is considered a more general form for an exponential function. Here, a is 3, and b is 2. Points on the graph include $(-2, 0.75)$, $(-1, 1.5)$, $(0, 3)$, $(1, 6)$, $(2, 12)$, and so on:

| | | | | | | | | |
|----------|------|-----|---|---|----|----|----|----|
| x | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
| y | 0.75 | 1.5 | 3 | 6 | 12 | 24 | 48 | 96 |

You may recognize a pattern in the y-coordinates.

Notice from both the table and the graph that the curve has y-intercept (0, 3); otherwise the curve is similar to $y = 2^x$ in terms of rate of increase and in terms of its horizontal asymptote ($y = 0$).



The form $y = a \cdot b^x$ is considered a more general form for an exponential function. Until the function $y = 3 \cdot 2^x$, the value of a has been 1 on every example. In general, $(0, a)$ represents the y-intercept of the graph. The value of a affects a vertical stretch of the curve. The value of b affects whether the graph increases or decreases and general steepness of the graph.

All the transformations also apply to these general building block exponential functions. So, $y = -2^{x+5} - 6$ would be the graph of $y = 2^x$ shifted left 5 units, reflected over the x-axis, and then shifted down 6 units. For more on these and other transformations, revisit Section 4.2.

One final technology note: When using the graphing calculator for exponential regression (“ExpReg”), the calculator computes the constants a and b for the function $y = a \cdot b^x$.

The exponential regression form used by Microsoft Excel is $y = ae^{kx}$, and the computer software calculates the constants a and k . Comparing calculator and computer regression equations, the values for a should be identical, and the values for b and k are very clearly related. The constant k is known as the growth rate (if $k > 0$) or the rate of decay (if $k < 0$).

Exercises:

For #1–6, create a table of values and graph the given exponential functions. Be sure to show any intercepts and asymptotes in your graph.

- $y = 3^x$
- $y = \left(\frac{1}{4}\right)^x$
- $y = 0.5^x$
- $y = 7^x$
- $y = 2\left(\frac{3}{4}\right)^x$
- $y = 3 \cdot 5^x$

Describe in a sentence the relationship between the graph of $y = 2^x$ and the graph of each of the following exponential functions.

7. $y = 2^{x-3} + 1$

8. $y = 2^{x+4} - 2$

9. $y = 2^{-x}$

10. Find any x- and y-intercepts of the graph of $y = 2^{x+4} - 2$.

11. Find any x- and y-intercepts of the graph of $y = 2^{x-3} + 1$.

12. Using transformations, sketch the graph of $y = -2^{x-3} - 4$.

13. Using transformations, sketch the graph of $y = \left(\frac{1}{2}\right)^{x+2} + 4$.

14. If each of the tables below represent points on the graphs of exponential functions, find the missing table values and the equation for each exponential function.

(a)

| x | y |
|----|-----|
| -2 | |
| -1 | 0.1 |
| 0 | 1 |
| 1 | 10 |
| 2 | 100 |
| 3 | |

*(b)

| x | y |
|----|-----|
| -2 | |
| -1 | 20 |
| 0 | 10 |
| 1 | 5 |
| 2 | 2.5 |
| 3 | |

*(c)

| x | y |
|----|-----|
| -2 | 2 |
| -1 | |
| 0 | 18 |
| 1 | 54 |
| 2 | 162 |
| 3 | |

15. List 2 features that graphs of the following exponential functions have in common.

$$y = 5^x, y = 100^x, y = 0.0033^x, \text{ and } y = \left(\frac{15}{16}\right)^x$$

*16. Find the remainder when 2^{625} is divided by 3. [Hint: Use smaller powers of 2 and look for a pattern in the remainders.]

*17. Find the remainder when 2^{72} is divided by 3. [Hint: Use smaller powers of 2 and look for a pattern in the remainders.]

*18. Simplify the following expression: $\frac{2^{1000} + 2^{997}}{2^{1000} - 2^{997}}$ [Hint: Factor out the GCD for the numerator and denominator.]