Ratios, Proportions, and Percent Applications

The use of proportional thinking is a very important problem solving strategy, and it is at the core of many important mathematical concepts including slope, speed, percent, probability, similarity, scaling, direct and inverse variations, and trigonometric ratios.

Research indicates "a strong positive correlation between students' understanding of fractions and their overall success in mathematics" (Gomez, 2009), and "a foundational skill essential to success with algebra" (National Mathematics Advisory Panel, 2008).

There are 4 types of ratios:

(1) Part-to-part ratios	For example, slope is the ratio of "rise" to "run". A slope of ½ means a vertical change of 1 unit for every horizontal change of 2 units.
(2) Part-whole ratios	Percents and probabilities tend to be part-whole ratios. 7% means 7 parts out of a total of 100.
(3) Ratios as quotients	For example, \$1.00 to 4 kiwis or \$3.00 for a dozen kiwis
(4) Ratios as rates	For example, the ratio above is equivalent to \$0.25 per kiwi.

Developing understanding of and applying proportional relationships is a focus of grade 7 (CCSSO, 2010; NCTM, 2006). But reasoning proportionally doesn't begin in middle school; multiplicative reasoning and fractional concepts start much earlier.

Proportional thinking involves understanding ratios as distinct entities, recognizing proportional relationships, having a sense of covariation (how quantities vary together), and developing a wide variety of strategies for solving proportions or comparing ratios. One of the strategies for solving proportions involves setting cross products equal.

> The cross products principle states that, if $\frac{a}{b} = \frac{c}{d}$, then $a \cdot d = b \cdot c$.
>
> For instance, $\frac{5}{8} = \frac{x}{100}$ implies that $8 \cdot x = 5 \cdot 100$, which leads to x = 62.5.

<u>Examples</u>:

(1) Jack and Jill were at the bottom of a hill, each hoping to fetch a pail of water. Jack walks uphill at 5 steps every 25 seconds, while Jill walks uphill at 3 steps every 10 seconds. Assuming a constant walking rate, who will get to the pail of water first?

Solution: This seems to be a good time to use the least common multiple of the two times given (25 seconds and 10 seconds), which is 50 seconds.

Jack's rate is $\frac{5 \text{ steps}}{25 \text{ seconds}} \times \frac{2}{2} = \frac{10 \text{ steps}}{50 \text{ seconds}}$

Jill's rate is $\frac{3 \text{ steps}}{10 \text{ seconds}} \times \frac{5}{5} = \frac{15 \text{ steps}}{50 \text{ seconds}}$

Since Jill goes further (15 steps is greater than 10 steps) in 50 seconds, Jill fetches the pail of water first.

(2) Some of the hens in Farmer Brown's chicken farm lay brown eggs, and some lay white eggs. Farmer Brown noticed that in the old hen house, she collected 4 brown eggs for every 10 white eggs. In the new hen house, the ratio of brown eggs to white eggs was 1 to 3. If both hen houses produced the same total number of eggs, in which hen house will there be more brown eggs?

Solution:

Using a brown to white ratio approach, we are comparing 4/10 with 1/3.

Using the LCD (30 white eggs), we have

$$\frac{4 \text{ brown eggs}}{10 \text{ white eggs}} \times \frac{3}{3} = \frac{12 \text{ brown eggs}}{30 \text{ white eggs}} \quad \text{vs.} \quad \frac{1 \text{ brown eggs}}{3 \text{ white eggs}} \times \frac{10}{10} = \frac{10 \text{ brown eggs}}{30 \text{ white eggs}}$$

The hen house with the 1-to-3 brown-to-white egg ratio would have more brown eggs.

(3) A black Labrador named Esther weighs 62 lb. If 1 kg = 2.20 lb, convert Esther's weight to kilograms. Round to the nearest kilogram.

Solution:

This can well be handled using a unit analysis approach.

$$62 \text{ lb} \cdot \frac{1 \text{ kg}}{2.20 \text{ lb}} \approx 28 \text{ kg}$$

A proportional thinking approach would also work well.

$$\frac{1 \text{ kg}}{2.20 \text{ lb}} = \frac{x}{62 \text{ lb}}$$

$$2.20 \cdot x = 62$$

$$x = \frac{62}{2.20} \approx 28 \text{ kg}$$

(4) "The Paper Stack Problem":

A ream of paper (which contains a stack of 500 pieces of paper) stands 2 inches tall. How tall would a stack of 6 billion pieces of paper be? Give your answer in

(a) Inches.

Solution:

Write the proportion. $$\frac{500 \text{ pieces}}{2 \text{ inches}} = \frac{6{,}000{,}000{,}000 \text{ pieces}}{x \text{ inches}}$$

Set cross products equal. $500 \cdot x = 2 \cdot 6{,}000{,}000{,}000$

Then perform basic algebra. $x = \dfrac{12{,}000{,}000{,}000}{500} = 24{,}000{,}000$ inches

(b) Feet. $24{,}000{,}000 \text{ inches} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} = 2{,}000{,}000 \text{ feet}$

(c) Miles. (Round to the nearest tenth of a mile.)

$$2{,}000{,}000 \text{ ft} \cdot \frac{1 \text{mi}}{5280 \text{ ft}} = 378.8 \text{ mi}$$

Direct Variation

Consider this example. A chef earns $23 an hour. In 1 hour, $23 is earned; in 2 hours, $46 is earned; in 3 hours, $69 is earned; and so on. Here are the results:

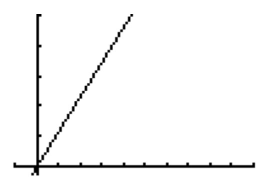
Time worked	1 hour	2 hours	3 hours	. . .	n hours
Total pay	$23	$46	$69	. . .	$23·n

Notice that the ratio of the total pay to the time worked is the same number for each pair.

$$\frac{\$23}{1 \text{ hour}} = \frac{\$46}{2 \text{ hours}} = \frac{\$69}{3 \text{ hours}} = \ldots = \frac{\$23n}{n \text{ hours}}$$

Whenever a situation produces pairs of numbers in which the ratios are constant as with this chef salary scenario, this application is called a **direct variation**. In this example, total pay **varies directly** with time worked, or in other words, total pay is **directly proportional** to time worked. As time worked increases, total pay increases proportionally.

For the chef pay example, $y = 23 \cdot n$, where y represents total pay in dollars and x represents time worked in hours. The graph of this equation would be a line through (0, 0) and with a steep slope. The greater the hourly pay, the greater the steepness of the line.



All direct variations have the form $y = k \cdot x$,
where k is the **constant of proportionality**.

Manipulating this formula yields the more useful equation
$y = \dfrac{k}{x}$ which we will study in greater depth. For two particular ordered pairs (x_1, y_1) and (x_2, y_2), we have $\dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$. Many realistic applications can be handled with a simple proportion.

Other examples involving direct variation abound!

- For a car cruising down the highway at 70 mph, distance varies directly with time. $[\, d = 70 \cdot t \,]$

- The circumference of a circle varies directly with the radius. [$C = 2\pi\pi$]

- For a hydraulic press, the input force on a small piston varies directly as the output force on the large piston (giving a mechanical advantage).

- The area of a circle varies directly with the square of the radius. [$A = \pi r^2$]
  This is a little more complicated because of the squaring; this is a quadratic function.

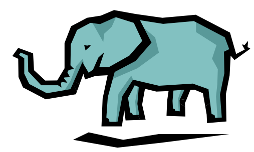
- Your grade in a class varies directly with your effort and attendance!

(5) The weight of an object on Earth varies directly with the object's weight on Pluto. A 100-lb object on Earth weighs 3 lb on Pluto. If an elephant weighs 4,100 lb on Earth, find the elephant's weight on Pluto.

Solution: $$\frac{100 \text{ lb}}{3 \text{ lb}} = \frac{4{,}100 \text{ lb}}{x \text{ lb}}$$

Since $100 \text{ lb} \cdot 41 = 4{,}100 \text{ lb}$,

$3 \text{ lb} \cdot 41 = 123 \text{ lb}$



From a basketball player's free throw percentage to current interest rates, from nutrition labeling to stock market trends, commission, and computing taxes — practical applications involving percents are extremely common in everyday life!

What does percent mean? The word *percent* means *per hundred*. For example, an interest rate of 5% means 5 cents on every 100 cents (or dollar) invested. A free throw percentage of 80% means that the basketball player makes 80 free throws in 100 attempts, or 4 out of 5.

<u>Writing a Percent as a Decimal</u>:
Drop the percent symbol and move the decimal point 2 places to the left.

E.g., 35% = 35 hundredths = 0.35, 83.5% = 0.835, 250% = 2.50

<u>Writing a Decimal as a Percent</u>:
Move the decimal point 2 places to the right and attach the percent symbol.

E.g., 0.25 = 25 hundredths = 25%, 0.715 = 71.5%, 0.2 = 0.20 = 20%

<u>Writing a Fraction as a Percent</u>:
Make the fraction into a decimal first, then move the decimal point 2 places to the right and attach the percent symbol OR solve a proportion using cross products.

E.g., $\frac{3}{8} = 0.375 = 37.5\%$ OR $\frac{3}{8} = \frac{p}{100} \Rightarrow 8 \cdot p = 300 \Rightarrow p = 37.5$

<u>Writing a Percent as a Fraction</u>:
Write the number before the '%" sign in the numerator, with 100 in the denominator, and then simplify, or multiply the number before the '%" sign by $\frac{1}{100}$.

E.g., $6\% = \frac{6}{100} \div \frac{2}{2} = \frac{3}{50}$ $\qquad$ $2\frac{1}{4}\% = \frac{9}{4} \cdot \frac{1}{100} = \frac{9}{400}$

Many percent applications fall under one of 4 problem categories. Three of these four involve the terms: part, rate, and base. We generally recommend using the formula approach to problem solving with percents. The base and the part should have the same measurement units.

Symbol/Meaning	**To find, use the formula . . .**
P = usually some fractional PART of the base	$P = B \cdot R$
R = the RATE (percent)	$R = \frac{P}{B}$
B = the BASE (the whole or entire amount)	$B = \frac{P}{R}$

<u>Examples</u>:

(6) Finding the part [This is the most common type of application.]

The Atlanta Braves won 84% of the games they played in June. If they played 25 games, how many did they win? GO BRAVES!

Solution: The Braves won 84% of the 25 games, so we're finding the part.

$P = B \cdot R = 25 \text{ games} \cdot 84\% = 25 \text{ games} \cdot 0.84 = 21 \text{ games}$

(7) Finding the rate

On the last test, 25 out of 30 students earned passing grades. What percent of the students passed?

Solution: Clearly, we have the part (25) and the base (30) given.

$R = \frac{P}{B} = \frac{25 \text{ students}}{30 \text{ students}} = 0.833...$ B 83%

(8) Finding the base

Aluminum is 12% of the mass of a certain truck. This car has 360 lb of aluminum in it. What is the total mass of the truck?

Solution: 360 lb is 12% of the total mass of the truck (the base)

$B = \frac{P}{R} = \frac{360 \text{ lb}}{12\%} = \frac{360 \text{ lb}}{0.12} = 3{,}000 \text{ lb}$

(9) Finding the percent change (increase or decrease)

This extremely common application category involves comparing the amount of change to the original amount, and then expressing this fraction as a percent. [Note: Subtract to find the "amount of change", and be careful as you notice the "original value" (the first, starting, or base amount).]

$$\text{Percent change} = \frac{\text{Amount of change}}{\text{Original amount}} \cdot 100\%$$

Enrollment increased from 2631 students to 2857 students over the last full academic year. Find the percent of increase in enrollment.

Solution: $\text{Percent change} = \frac{2{,}857 - 2{,}631}{2{,}631} \cdot 100\% = \frac{226}{2{,}631} \cdot 100\% = 8.6\% \text{ or } 9\%$

(10) Ricky made 7 out of 12 free throws during a recent basketball game. What percent did he make?

$\frac{7}{12} = \frac{P}{100}$ Set the cross products equal (or multiply both sides by 1200).

$7 \times 100 = 12 \times P$ Then divide 700 by 12.

$\frac{700}{12} = 58.\overline{3}$ Ricky made roughly 58% of his free throws. He needs to practice!

(11) A dress regularly sells for $45. If it is on sale for 20% off, what is the discount? What is the sale price?

Find 20% of 45.	$0.20 \times 45 = 9$	The discount is $9.
Then subtract the discount.	$45 - 9 = 36$	The sale price is $36.

(12) The median cost of housing rose 20% in one city in a particular year. The median price was then $240,000. What was the median cost of housing at the end of the previous year?

Let x be the original median cost. The increase would be $0.20 \cdot x$

The original cost plus the increase would be the new median price.

$$x + 0.20 \cdot x = 240000$$

Combine the 1.00x and the 0.20x. $1.20 \cdot x = 240000$

Then divide 240,000 by 1.20. $x = 200000$

The median cost was $200,000.

Ratios, Proportions, and Percent Applications Worksheet Name __________________

Show work to support each answer you give. Your reasoning <u>process</u> is more important than the answer!

Solve the following proportions.

1. $\frac{3}{8} = \frac{63}{x}$

2. $\frac{7}{12} = \frac{p}{100}$

*3. $\frac{x+1}{x+2} = \frac{5}{3}$

4. Terry can run 4 laps in 12 minutes. Susan can run 3 laps in 9 minutes. Who is the faster runner?

5. One class has 10 girls and 15 boys, and another class has 9 girls and 12 boys. Which class has the higher ratio of girls to total students?

6. Fred had 16 questions correct out of 20 on a recent proportional reasoning quiz, and Ethyl answered 42 correct out of 50 on an algebraic thinking test. Who performed better?

7. Which mixture would taste more “orangey”: 1 can of concentrate with 3 cans of water or 2 cans of concentrate with 6 cans of water? Multiple choice. Explain your reasoning.

(a) 1 can of concentrate with 3 cans of water
(b) 2 cans of concentrate with 6 cans of water
(c) They would taste the same.

8. Hooke’s law states that the distance d that a spring will stretch varies directly as the mass m of an object hanging from the spring. If a 2-kg mass stretches a spring 4 cm, how far will a 3-kg mass stretch the spring? Round to the nearest centimeter.

9. The Play-A-Lot Video Game Store charges $2.00 for every 15 minutes to play on their wide selection of video games. Wired-For-Action Video Store charges $3.00 for 20 minutes of play on their video games. Where would you choose to go if you make your decision based on pricing?

10. To the nearest mile, how tall would a stack of 1 trillion pieces of paper reach? [Hint: Use the fact that a ream of paper (which contains a stack of 500 pieces of paper) stands 2 inches tall.]

11. Given: A roll of 50 pennies stands 3 inches high. Round all three solutions to the nearest tenth of a mile.

(a) How tall would a stack of 1 million pennies be?

(b) How tall would a stack of 2 billion pennies be?

12. In a recent year, approximately 1.7 million college students received bachelor's degrees. Express this number of students in terms of the number of football stadiums (capacity 70,000) that could be filled by these graduates.

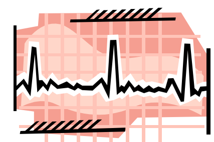
13. The scale on a map is $\frac{1}{4}\text{in} : 10\text{ mi}$. If two cities are $1\frac{1}{2}$ in. apart on the map, what is the actual distance between them?

14. In *Gulliver's Travels*, Gulliver traveled to Brobdingnag, where people are 10 times as tall as normal people, and to Lilliput, where people are $\frac{1}{10}$ as tall as normal people. Les Moore, a normal person of average build, is 176 cm tall. Estimate the height of

(a) A Brobdingnagian (Use 2 different metric units.)

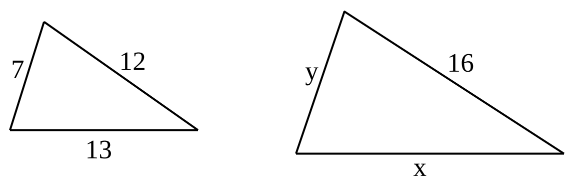
(b) A Lilliputian (Use 2 different metric units.).

15. Approximately 720,000 Americans died of a heart attack in a recent year. Express this quantity in heart attacks per day. [Use 365 days for 1 year.]



16. To estimate the number of people in Jackson, population 50,000, who have had a flu shot this year, 250 people were surveyed. Of those polled, 27 had a flu shot. How many people might we expect to have had a flu shot in this town?

17. When triangles are similar, corresponding sides are proportional. For the following similar triangles, find x and y.



18. There were approximately 132 million births worldwide in a recent year. Express this quantity in births per minute.

19. Mariano Rivera was a "closer" for the New York Yankees. In his 19-year career as a major league pitches, Rivera earned a record 652 saves in 732 attempts. What percent of his save opportunities were successful?

20. During a season, a little league baseball team wins 12 games and loses 7 games.

(a) What percent have they won?

(b) If they were a MLB baseball team and continued this pace, predict how many games they will win in a full 162-game season.

21. Kathy owed her friend $80 and then paid off $36 from her tip earnings.

(a) What percent of the original debt had she paid?

(b) What was her new debt after the payment?

22. Fred makes 7 out of 12 free throws in the NCAA tournament. What percent did he make?

23. Gala apples regularly sell for $1.59 per pound. If they go on sale for 30% off the regular price, what is the sale price?

24. A dress regularly sells for $45. If it is on sale for 20% off, what is the discount? What is the sale price?

25. Write your own proportion or percent application problem. Be creative!