Relative properties of reals

by

Bernard August Anderson

B.A. (Yale University) 1999

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

GRADUATE DIVISION
of the
UNIVERSITY of CALIFORNIA at BERKELEY

Committee in charge:

Professor Theodore A. Slaman, Chair
Professor W. Hugh Woodin
Professor Daniel Warren

Spring 2007

The dissertation of Bernard August Anderson is approved:

Chair Date

Date

Date

University of California at Berkeley

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Abstract

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This paper examines several properties of reals in some relative context. We consider in detail reals which are relatively recursively enumerable, reals which are n-generic relative to some perfect tree, and reals which are relatively hyperimmune-free. We seek to classify which reals hold these properties and study the implications of certain reals being included or excluded.

Many of the findings are unexpected. All but countably many reals are n-generic relative to some perfect tree and relatively hyperimmune-free. However much of the hierarchy of iterated hyperjumps do not hold these properties. Indeed, for genericity we need ZFC^- and infinitely many iterates of the power set of ω to complete the proof. The set of relatively recursively enumerable reals is, in some sense, as large as possible. However every nonempty Π^0_1 class and the set of relatively REA reals each contain a real which is not relatively recursively enumerable.

We say that a real X is relatively r.e. if there exists a real Y such that X is r.e. (Y) and $X \not\leq_T Y$. We say X is relatively REA if there exists such a $Y \leq_T X$. We define $A \leq_{e_1} B$ if there exists a Σ_1 set C such that $n \in A$ if and only if there is a finite $E \subseteq B$ with $(n, E) \in C$. We show that a real X is relatively r.e. if and only if $\overline{X} \not\leq_{e_1} X$.

We prove that every nonempty Π^0_1 class contains a real which is not relatively r.e. We also construct a real which is relatively r.e. but not relatively REA. We show that for all reals X and Z such that $\overline{X} \not\leq_{e_1} X$, $X <_T Z$, and Z is REA (X) there is a real Y such that $Y \leq_T Z$, $Y \not\leq_T X$, and X is r.e. (Y). We also show that for every real X such that $\overline{X} \not\leq_{e_1} X$ there is a Y such that X is r.e. (Y), $X \not\leq_T Y$, and Y is not arithmetic in X.

A real X is relatively simple and above if there exists a real Y such that X is r.e. (Y) and there is no infinite $Z \subseteq \overline{X}$ such that Z is r.e. (Y). We prove that every 1-generic real is relatively simple and above.

We say that a real X is n-generic relative to a perfect tree T if X is a path through T and for all $\Sigma^0_n(T)$ sets S, there exists a number k such that either $X|k \in S$ or for all $\sigma \in T$ extending $X|k$ we have $\sigma \notin S$. A real X is n-generic relative to some perfect tree if there exists such a T.

We first show that for every number n all but countably many reals are n-generic relative to some perfect tree. Second, we show that proving this statement requires ZFC$^-$ + "$\exists$ infinitely many iterates of the power set of ω". Third, we prove that every finite iterate of the hyperjump, $\mathcal{O}^{(n)}$, is not 2-generic relative to any perfect tree and for every ordinal α below the least λ such that $\sup_{\beta<\lambda}(\beta\text{th admissible}) = \lambda$, the iterated hyperjump $\mathcal{O}^{(\alpha)}$ is not 5-generic relative to any perfect tree.

We show that no ranked real is 1-generic relative to some perfect tree. We prove that a 2-generic real cannot compute a nonrecursive ranked real. We note that no real of n-REA degree for $n \in \omega$ is 1-generic relative to some perfect tree but construct a 1-generic real which is ω-r.e.

We say that a real X is relatively hyperimmune-free if there exists a real Y such that $X \not\leq_T Y$ and for every function $f \leq_T X \oplus Y$ there is a function $g \leq_T Y$ which dominates f. We prove that all but countably many reals are relatively hyperimmune-free. We demonstrate that reals of α-REA degree are not relatively hyperimmune-free for all $\alpha < \omega_1^{CK}$. Finally, we show that for every ordinal α below the least λ such that $\sup_{\beta<\lambda}(\beta$th admissible$) = \lambda$, the iterated hyperjump $\mathcal{O}^{(\alpha)}$ is not relatively hyperimmune-free.

Professor Theodore A. Slaman
Dissertation Committee Chair

To God, who my life is dedicated to.

Contents

Acknowledgements

Few students receive a Ph. D. without significant help from many other people and I believe I am indebted more than most. I would first like to thank my advisor, Ted Slaman. He is a dedicated mentor, an excellent teacher, and one of the nicest people I have known. Leo Harrington and Hugh Woodin were always happy to help and to answer any questions that I had. I would also like to thank Tom Scanlon and John Steel who shared their knowledge and appreciation of logic with me, Ole Hald who taught me how to teach, and Daniel Warren who served as the outside member of my dissertation and qualifying committees.

I could not have navigated Berkeley's bureaucracy without the help of the Mathematics department staff, especially Thomas Brown, Linda Jarvis, Mike Kim, Dave Mina, Barbara Peavy, Marsha Snow, and Barb Waller. I was financially supported by the National Science Foundation (grant # DMS-0501167), the citizens of California, and the National University of Singapore Institute of Mathematical Sciences. Johanna Franklin, Jeremy Sain, Tony Chiang, and many other graduate student colleagues made my time here easier (and more fun). I also wish to thank my students, who always remind me why I love to teach.

I am heavily indebted to the community at Newman Hall Holy Spirit who helped me to grow as a person and taught me the meaning of friendship. There are far too many individuals to thank (despite my efforts below).

Most of all, I am grateful for my family. The constant love and support of my parents Gai and Robert and my brothers and sisters (in-law) Ben, David, Randy, Tina, and Todd guided me through storms I could not otherwise have weathered. I am also thankful for my grandparents Bernard and Marcella, my nephews and niece Christian, Gabriella, Jacob, and Weston and the rest of my extended family who have added joy to my life.

Finally, I wish to thank the following people not yet named:

Uri Andrews, Michael Barnett, Eric Bellm, Fr. Andrew Bobola, Adam Booth, Michael Constant, Jamie Cortes, Brandon de la Cruz, Archie Davenport, Mai Dinh, Fr. Charlie Donahue, Ken Easwaran, Lexi Ford, Jecy Furlani, Peter Gerdes, John Goodrick, David Hernandez, Alan Hsia, Noelle Koleszar, Phuoc Kong, Katherine Lao, Fr. Paul Mariani, Alice Medvedev, David Morales, Kathleen Natividad, Ryan Paul Nebres, Delwin Ng, Rev. Pius Ojara, Tommy Paderes, Chris Page, Fransesca Po, Aaron Rios, Amy Robbins, Amanda Stempson, John Tsou, Judy Tuan, Tim Wescott, and James Yao.

Chapter 1

Introduction

In this thesis we will study the behavior of reals, which we view as elements of the Cantor space (2^ω), infinite binary strings. While we will use the Turing degrees of the reals, we are mainly concerned with the reals themselves. We examine three properties of reals: recursive enumerability, genericity, and nonhyperimmunity. In each case, there are computational bounds on which reals can hold these properties. For example, reals strictly above $0'$ hold none of them. However, characteristics of these properties can still appear when working with reals outside these bounds.

We wish to answer the question: Which reals hold these properties in some appropriate relative context, and which inherently lack them? In particular, we want to determine which reals are relatively recursively enumerable, which reals are n-generic relative to some perfect tree, and which reals are relatively hyperimmune-free. In a sense this is an inversion problem, since we try to find for which reals X is there another real or a perfect tree which witnesses X holding the relative property. We find that the sets of reals holding these relative properties are surprisingly large, yet there are many interesting reals which are not contained in them. Furthermore, we observe that our analysis of which reals are n-generic relative to some perfect tree requires an unexpectedly large fragment of ZFC.

We consider separately each of these properties in more detail below. We note that much of the material in this thesis is expected to appear in similar or identical form in two currently unpublished articles by the author [1, 2].

Relatively recursively enumerable reals

The study of recursively enumerable (r.e.) reals is a fundamental part of Recursion Theory. A real is r.e. if it is the range of a recursive function; there is an effective procedure to list its elements. We want to determine when a real is r.e. relative to another real without being recursive in it.

Definition 1. A real X is relatively r.e. if there exists a real Y such that X is r.e. (Y) and $X \not\leq_T Y$.

Definition 2. A real X is relatively REA if there exists a real $Y \leq_T X$ such that X is r.e. (Y) and $X \not\leq_T Y$.

Jockusch first made progress towards classifying the relatively r.e. reals by showing that all 1-generic reals are relatively REA [6]. Later, Kurtz proved that the set of relatively REA reals has measure one [12]. Kautz improved this by demonstrating that all 2-random reals are relatively REA [10]. While the set of relatively r.e. reals seems very large, there is a natural limit to its size. Given any real X, we can find a real Y in the same Turing degree such that Y is not relatively r.e. We simply let Y be the set of initial segments of

X. More generally, a real X is not relatively r.e. any time there is a Σ_1 machine taking enumerations of X to enumerations of $\overline{X}$.

We will show that this is the only case in which X is not relatively r.e. We begin with some definitions of reductions where one enumeration is computed from another.

Definition 3. $A \leq_{e_n} B$ if there is a Σ_n set C such that $n \in A$ if and only if there is a finite $E \subseteq B$ with $(n, E) \in C$.

Definition 4. $A \leq_e B$ if there is a Σ_1 machine which, given an enumeration of B in any order, outputs an enumeration of A.

It is easy to see that $A \leq_{e_1} B$ iff $A \leq_e B$ and we will use the terms interchangeably from now on. We will prove that a real X is relatively r.e. if and only if $\overline{X} \not\leq_e X$. This demonstrates that the set of relatively r.e. reals is, in some sense, as large as possible.

We will show that not every relatively r.e. real is relatively REA by a direct construction. While we cannot guarantee that an X such that $\overline{X} \not\leq_e X$ is relatively REA in the general case, we can always find a witness Y that is close to being recursive in X. Given any $Z >_T X$ where Z is REA (X), we can find $Y \leq_T Z$.

Jockusch and Soare showed that every nonempty Π^0_1 class contains a real which has hyperimmune-free degree and hence is not relatively REA [9]. We sharpen this by showing every nonempty Π^0_1 class contains a real which is not relatively r.e. We also offer an improvement to Jockusch's result that all 1-generic reals are relatively REA [6].

Definition 5. A real X is relatively simple and above if there exists a real $Y \leq_T X$ such that X is r.e. (Y) and there is no infinite $Z \subseteq \overline{X}$ such that Z is r.e. (Y).

We will prove that all 1-generic reals are relatively simple and above.

Reals n-generic relative to some perfect tree

A real is n-generic if for every Σ^0_n set there is an initial segment of the real which either meets the set or for which no extension of the segment can meet the set. These reals have many interesting characteristics and have been studied extensively (see Jockusch and Posner [7] and Kumabe [11] among others). While the set of n-generics is comeager, it is in some ways limited. In particular, it is completely excluded from the cone above $\mathbf{0}'$ since no 1-generic can compute a nonrecursive r.e. set.

We wish to determine how this set might be expanded from reals which are n-generic to those that can be made to seem n-generic in some appropriate context. An attractive framework for this question is to consider reals which are n-generic when viewed as paths through a given perfect tree, rather than all of 2^ω.

Definition 6. A real X is n-generic relative to a perfect tree T if X is a path through T and for all $\Sigma^0_n(T)$ sets S, there is a k such that either $X|k \in S$ or $\sigma \notin S$ for every $\sigma \in T$ extending $X|k$.

Definition 7. A real X is n-generic relative to some perfect tree if there exists a perfect tree T such that X is n-generic relative to T.

We show that the set of reals not n-generic relative to any perfect tree is countable. From this we can infer that many reals with properties that are not normally associated with genericity still seem generic in the context of some perfect tree. For example, there are reals of minimal degree and reals with high information content, such as the theory of second order arithmetic, that are generic relative to some perfect tree.

The proof that the set of reals not n-generic relative to any perfect tree is countable uses ZFC$^-$ and n iterates of the power set of ω. We show that for sufficiently large n, this requirement is sharp and cannot be significantly improved. From this we see that for reasonably high values of n, the set of reals not n-generic relative to any perfect tree is unusually large for a countable set of this type. It provides a natural example of a set which needs this level of ZFC to be understood.

The set of reals not n-generic relative to some perfect tree behaves similarly for low values of n. By looking at the iterates of the hyperjump, we demonstrate that the set still contains reals of unexpectedly high complexity. Even for $n = 2$, relatively large fragments of arithmetic fail to prove the set is countable. We also begin to characterize the sets that are 1-generic relative to some perfect tree.

In a similar vein, Reimann and Slaman [17] have recently studied the set of reals which appear random in some context, in this case relative to some continuous measure. Our results for genericity are analogous to what they have discovered for randomness in surprisingly many, but not all, instances.

Relatively hyperimmune-free reals

We also find conclusions similar to those of Reimann and Slaman [17] in our study of relatively hyperimmune-free reals. A real is hyperimmune-free if every function it computes can be dominated by a recursive function. These reals have been well studied (see Martin and Miller [15] and Soare [21] among others). We note that for any real X if there exists a real Y such that $Y <_T X \leq_T Y'$ then X is not hyperimmune-free. This implies that no real which computes $0'$ or can be computed by $0'$ is hyperimmune-free. We seek to study which reals appear hyperimmune-free relative to another real.

Definition 8. A real X is relatively hyperimmune-free if there exists a real Y with $X \not\leq_T Y$ such that for every function $f \leq_T X \oplus Y$ there is a function $g \leq_T Y$ which dominates f.

There are several constructions of a nontrivial hyperimmune-free real which can be relativized to show that relatively hyperimmune-free reals are cofinal in the Turing degrees [9] [15]. For example, if X is a Spector minimal cover of Y then Y witnesses that X is relatively hyperimmune-free. Hence a Spector minimal cover of $0'$ is an example of a real which is relatively hyperimmune-free but not hyperimmune-free.

We show that the set of reals which are not relatively hyperimmune-free is countable. However, as with genericity, iterated hyperjumps provide examples of reals of high complexity in this set. At lower levels, we note that reals of α-REA degree are not relatively hyperimmune-free for all $\alpha < \omega_1^{CK}$.

Chapter 2

Relatively recursively enumerable reals

2.1 Main Theorem

Let X be a real such that $\overline{X} \not\leq_e X$. To show that X is relatively r.e., we will use a witness Y which is simply a list of the elements of X (viewed as a set). We use a monadic conversion function $m : \omega^\omega \to 2^\omega$, defined by $m(A) = 1^{A(0)}\hat{}0\hat{}1^{A(1)}\hat{}0\hat{}1^{A(2)}\hat{}0\ldots$. By definition, X is then r.e. (Y), but we must find an order for this list such that $Y \not\geq_T X$. We will do this using the partial order of all finite strings of elements of X. We choose Y to be the monadic form of a generic for this partial order.

Theorem 2.1.1. *X is relatively r.e. if and only if $\overline{X} \not\leq_e X$.*

Proof. ($\Longrightarrow$) Let $Y \not\geq_T X$ be such that X is r.e. (Y). Suppose $\overline{X} \leq_e X$. Then given Y we can enumerate X and use this to enumerate $\overline{X}$. Hence $\overline{X}$ is r.e. (Y), so $X \leq_T Y$ for a contradiction. Thus $\overline{X} \not\leq_e X$.

($\Longleftarrow$) Let $\mathbb{P}$ be the partial order $\mathbb{P} = \{\sigma \in \omega^{<\omega} \mid \forall n < \text{length}(\sigma)\,[\sigma(n) \in X]\}$, ordered by reverse inclusion. Let G be a 1-generic (X) real in this partial order and let $Y = m(G)$. Then X is r.e. (Y), since $n \in X$ if and only if $0\hat{\ }1^n\hat{\ }0 \subseteq Y$ (adding n to Y is dense). It remains to show $\overline{X}$ is not r.e. (Y).

Suppose $\overline{X} = W_k^Y$ for some k. We will use genericity to show any enumeration of X extending some condition computes an enumeration of $\overline{X}$. This will imply $\overline{X} \leq_e X$ for a contradiction. Let $S = \{\sigma \in \mathbb{P} \mid \exists n \in X\,[n \in W_k^{m(\sigma)}]\}$. Then $G \notin S$ and G is 1-generic (X), so for some $q \in G$ we have $\forall r \leq_{\mathbb{P}} q\,[r \notin S]$. Let $Q = \{p \in \mathbb{P} \mid p \leq_{\mathbb{P}} q\}$.

Claim. *$n \in \overline{X}$ if and only if $\exists p \in Q\,[n \in W_k^{m(p)}]$.*

Proof. ($\Longleftarrow$) $p \notin S$ so for all $l \in X$ we have $l \notin W_k^{m(p)}$. Hence $n \in \overline{X}$.

($\Longrightarrow$) $n \in \overline{X}$ so $n \in W_k^Y$. Then for some $r \in G$ we have $n \in W_k^{m(r)}$ and $r \in Q$. □

Given an enumeration of X, we can generate an enumeration of Q by adding elements of X to q in all possible orders. We can then find an enumeration of $\overline{X}$ using the claim. Hence $\overline{X} \leq_e X$ for the desired contradiction. Therefore $\overline{X}$ is not r.e. (Y) and X is relatively r.e. □

The proof can also be done in an arithmetic context, using an ordinary 1-generic (X) real G, and letting $Y = \{\langle n,m\rangle \mid n \in X \ \wedge\ \langle n,m\rangle \in G\}$. In this context we see that we can compute a witness Y from X and any 1-generic (X) real. Hence, given any real Z which is properly REA (X), we can find a witness which is recursive in Z.

Remark 2.1.2. *Let Z be REA (X), $Z \not\leq_T X$, and $\overline{X} \not\leq_e X$. Then there exists $Y \leq_T Z$ such that X is r.e. (Y) and $X \not\leq_T Y$.*

We state two corollaries.

Corollary 2.1.3. *Let $n \in \omega$. $\overline{X} \not\leq_{e_n} X$ if and only if there exists Y such that X is $\Sigma_n(Y)$ and X is not $\Delta_n(Y)$.*

Proof. $(\Longleftarrow)$ X is $\Sigma_n(Y)$ and X is not $\Delta_n(Y)$ for some Y. If $\overline{X} \leq_{e_n} X$ then $\overline{X}$ is $\Sigma_n(Y)$ for a contradiction. Hence $\overline{X} \not\leq_{e_n} X$.

$(\Longrightarrow)$ $\overline{X} \not\leq_{e_n} X$. This implies $\overline{X} \not\leq_{e_n} X \oplus 0^{(n)}$ since the existence of a subset of $0^{(n)}$ with a Σ_n property is a Σ_n question. By the argument used in the proof of Theorem 2.1.1 we can conclude there is a Z such that $X \oplus 0^{(n)}$ is r.e. (Z) and $\overline{X}$ is not r.e. (Z). We note that $0^{(n)}$ is r.e. (Z) implies $0^{(n)} \leq_1 Z'$ so $0^{(n-1)} \leq_T Z$. By the Friedberg Inversion Theorem, let Y be such that $Y^{(n-1)} \equiv_T Z$. Then X is r.e. $(Y^{(n-1)})$ so X is $\Sigma_n(Y)$. Suppose $\overline{X}$ is $\Sigma_n(Y)$. Then $\overline{X}$ is r.e. $(Y^{(n-1)})$, so $\overline{X}$ is r.e. (Z) for a contradiction. Hence X is not $\Delta_n(Y)$. □

Corollary 2.1.4. *Let $\overline{X} \not\leq_e X$. Then there exists Y such that X is r.e. (Y) and $X \oplus Y \geq_T Y'$.*

Proof. We use Theorem 2.1.1 to find a real $Z \not\geq_T X$ such that X is r.e. (Z). We then apply the proof of the Posner-Robinson Theorem above Z to get a real $Y \geq_T Z$ with $X \oplus Y \equiv_T Y'$. Then X is r.e. (Y) since X is r.e. (Z) and $Z \leq_T Y$. □

2.2 Relatively r.e. but not Relatively REA

We find a real A which is relatively r.e. but not relatively REA using a proof similar to Lachlan's construction of a minimal real [13]. We follow his definitions. We say σ and τ are adjacent at m if $\sigma(m) = 0$, $\tau(m) = 1$, and for all $n \neq m$ we have $\sigma(n) = \tau(n)$. If $\sigma \subseteq \tau$ we let $\tau - \sigma$ denote the string γ such that $\tau = \sigma\hat{}\gamma$. We say σ and τ split for e if $\{e\}^\sigma(n)\downarrow \neq \{e\}^\tau(n)\downarrow$. We define the function tree $T[\sigma]$ by $(T[\sigma])(\tau) = T(\sigma\hat{}\tau)$.

Definition 9. A function tree T is a 1-tree if for every $\sigma \in 2^{<\omega}$ and $i = 0, 1$ we have $T(\sigma\hat{}0)$ adjacent to $T(\sigma\hat{}1)$ at length$(T(\sigma))$ and the string $T(\sigma\hat{}i) - T(\sigma)$ depends only on i and length(σ).

Definition 10. A function tree T is e-regular if for every σ we have $T(\sigma\hat{}0)$ and $T(\sigma\hat{}1)$ split for e.

We define a function $\mathcal{T}$: 1-trees $\times\, \omega \to$ 1-trees, by induction on the height of the tree. $(\mathcal{T}(T, e))(\langle\rangle) = T(\langle\rangle)$. At stage s we label the strings of length s as $\sigma_1, \sigma_2, \ldots, \sigma_{2^s}$ and let $\tau_0 = \langle\rangle$. We search inductively for $\tau_1, \tau_2, \ldots \tau_{2^s}$ such that for $i = 0, 1$ and all $j \leq 2^s$ we have $\tau_j \supseteq \tau_{j-1}$ and $(\mathcal{T}(T, e))(\sigma_j)\hat{}i\hat{}\tau_j$ is on T. For each j we look for the first appropriate γ, n, h such that

$$\{e\}_h^{(\mathcal{T}(T,e))(\sigma_j)\hat{}0\hat{}\gamma}(n)\downarrow \neq \{e\}_h^{(\mathcal{T}(T,e))(\sigma_j)\hat{}1\hat{}\gamma}(n)\downarrow$$

We then let $\tau_j = \gamma$. If a search does not halt then we let $\mathcal{T}(T, e)$ be undefined. Otherwise, we define $\mathcal{T}(T, e)$ at level $s + 1$ by for $i = 0, 1$ and $j \leq 2^s$ setting $(\mathcal{T}(T, e))(\sigma_j\hat{}i) = (\mathcal{T}(T, e))(\sigma_j)\hat{}i\hat{}\tau_{2^s}$.

We observe that if T is a recursive 1-tree and $\mathcal{T}(T, e)$ is defined then $\mathcal{T}(T, e)$ is an

e-regular recursive 1-tree and a subtree of T.

We will build A using a sequence of recursive 1-trees $\langle T_j \mid j \in \omega \rangle$ and strings $\langle A_j \mid j \in \omega \rangle$ such that $T_{j+1} \subseteq T_j$ and $A_{j+1} \supseteq A_j$ for all j. We maintain $A_j = T_j(\langle\rangle)$ and let $A = \bigcup_{j \in \omega} A_j$.

Lemma 2.2.1. *There exists a real which is relatively r.e. but not relatively REA.*

Proof. We wish to meet the following requirements.

R_e W_e does not witness $\overline{A} \leq_e A$.

$N_{\langle e,k \rangle}$ If $W_k^{\{e\}^A} = A$ then $A \leq_T \{e\}^A$.

We begin with T_0 =id and $A_0 = \langle\rangle$. We order the priorities $R_1, N_1, R_2, N_2, \ldots$ and use an injury free priority argument. At stage $s+1$ we act to meet the strongest priority requirement not yet satisfied. Let $l =$ length (A_s).

To meet the requirement R_e we check to see if there is a finite set E with $(l, E) \in W_e$ and a string σ on T_s such that for all $n \in E$ we have $\sigma(n) = 1$. If no such E exists then let $\gamma = \langle 0 \rangle$. Otherwise, let τ be minimal such that $\sigma \subseteq T_s(\tau)$ and let γ be such that $\gamma(0) = 1$ and $\gamma(n) = \tau(n)$ for all n with $0 < n <$length (τ).

We then let $T_{s+1} = T_s[\gamma]$ and $A_{s+1} = T_{s+1}(\langle\rangle)$ and label R_e as satisfied.

To meet the requirement $N_{\langle e,k \rangle}$ we check to see if $\mathcal{T}(T_s, e)$ is defined. If it is, we let $T_{s+1} = \mathcal{T}(T_s, e)$, $A_{s+1} = A_s$, and label $N_{\langle e,m \rangle}$ as satisfied for all m.

Otherwise, there are strings ν and τ such that $T_s(\nu)\hat{}0\hat{}\tau$ is on T_s and for all σ with $T_s(\nu)\hat{}0\hat{}\tau\hat{}\sigma$ on T_s and all n, h we fail to have

$$\{e\}_h^{T_s(\nu)\hat{}0\hat{}\tau\hat{}\sigma}(n)\downarrow \neq \{e\}_h^{T_s(\nu)\hat{}1\hat{}\tau\hat{}\sigma}(n)\downarrow$$

We then let $m =$ length $(T_s(\nu))$ and consider two cases.

Case 1: There is an $i = 0, 1$ and a σ such that letting $\mu = T_s(\nu)\hat{\ }i\hat{\ }\tau\hat{\ }\sigma$ we have μ is on T_s and $m \in W_k^{\{e\}^\mu}$. We then let $\mu_0 = T_s(\nu)\hat{\ }0\hat{\ }\tau\hat{\ }\sigma$ and let γ be minimal such that $\mu_0 \subseteq T_s(\gamma)$.

Case 2: Else. We then let γ be minimal such that $T_s(\nu)\hat{\ }1\hat{\ }\tau \subseteq T_s(\gamma)$.

In either case, we let $T_{s+1} = T_s[\gamma]$ and $A_{s+1} = T_{s+1}(\langle\rangle)$ and label $N_{\langle e,k\rangle}$ as satisfied.

This completes our construction. We now wish to verify that A is relatively r.e. and not relatively REA. Let $e \in \omega$ be arbitrary. Let s be the step at which R_e was satisfied and let $l =$ length (A_s). If $l \in A$ then at step s we found a finite subset E of A_{s+1} with $(l, E) \in W_e$. Similarly, if $l \notin A$ then there is no finite subset E of A such that $(l, E) \in W_e$. Hence l prevents W_e from witnessing $\overline{A} \leq_e A$. Thus $\overline{A} \not\leq_e A$ and by Theorem 2.1.1 we conclude that A is relatively r.e.

To show A is not relatively REA, let $e, k \in \omega$ be arbitrary. Let s be the step at which $N_{\langle e,k\rangle}$ was satisfied. We may assume $\{e\}^A$ is total since we are done if it is not. If $\mathcal{T}$ was defined at step s then T_{s+1} is e-regular and can be used to show $A \leq_T \{e\}^A$ by the usual minimality argument. If it was undefined, let m be as in step s and consider the two cases. If the second case applied then $m \in A$ but $m \notin W_k^{\{e\}^A}$. If the first case applied with $i = 0$ then $m \notin A$ but $m \in W_k^{\{e\}^A}$. Finally, if the first case applied with $i = 1$ let μ and μ_0 be as in step s and let u be such that $m \in W_k^{\{e\}^\mu | u}$. Then $\{e\}^\mu | u$ and $\{e\}^{\mu_0} | u$ both converge so they must be equal. This implies $m \in W_k^{\{e\}^{\mu_0}}$ so $m \in W_k^{\{e\}^A}$, but we have $m \notin A$.

Thus in all cases either $A \leq_T \{e\}^A$ or $A \neq W_k^{\{e\}^A}$. Therefore, A is not relatively REA. $\square$

2.3 Π^0_1 Classes

Theorem 2.3.1. *Every nonempty Π^0_1 class contains a real which is not relatively r.e.*

Proof. Let T be a recursive tree such that the members of the Π^0_1 class are the paths through T. We will inductively construct a real X which will be the rightmost path through T and a set C which will witness $\overline{X} \leq_{e_1} X$. The procedure will be recursive and we will only add elements to C so that C will be r.e.

We begin with $X_0 = \langle\rangle$ and $C_0 = \emptyset$. At stage $s+1$, if $X_s\hat{\ }1 \in T$ we let $X_{s+1} = X_s\hat{\ }1$ and $C_{s+1} = C_s$. Otherwise, let l be greatest such that $X_s|l\hat{\ }0 \in T$ (l must exist since the class is nonempty). We then let $X_{s+1} = X_s|l\hat{\ }0$ and $C_{s+1} = C_s \cup (l, \{n < l \mid X_s(n) = 1\})$.

We observe that X is the rightmost path through T (we set $X(m) = 0$ if and only if there is no path through T extending $X|m\hat{\ }1$). We wish to show for all $m \in \omega$ that $m \notin X$ if and only if $(m, E) \in C$ for some finite $E \subset X$ (viewed as a set). Suppose $m \notin X$. Let s be least such that for all $t > s$ we have $X_s|m = X_t|m$. Then $(m, \{n < m \mid X(n) = 1\})$ was added to C at stage $s+1$.

Conversely, suppose $E \subset X$ and (m, E) was added to C at stage s. We note that the value of X at n does not change from 0 to 1 unless the value of X at k changes from 1 to 0 for some $k < n$. As a result, if $X_s|m \neq X_t|m$ for some $t > s$ then $\{n < m \mid X_s(n) = 1\} \not\subseteq \{n < m \mid X(n) = 1\}$ so $E \not\subseteq X$. Thus $X_s|m = X|m$ and $m \notin X$. Therefore C witnesses $\overline{X} \leq_{e_1} X$ so by Theorem 2.1.1, X is not relatively r.e. $\square$

Given any property with a nonempty Π^0_1 class of reals holding the property we can apply Theorem 2.3.1 to find a real X with this property which is not relatively r.e.

Corollary 2.3.2. *There is a 1-random real which is not relatively r.e.*

Corollary 2.3.3. *There is a diagonally non-recursive (DNR) real which is not relatively r.e.*

Corollary 2.3.4. *There is a real coding a complete extension of Peano Arithmetic which is not relatively r.e.*

Corollary 2.3.5. *There is a Schnorr trivial real which is not relatively r.e.*

Corollary 2.3.2 contrasts with the result of Kautz that all 2-random reals are relatively REA [10]. We also note that X has r.e. degree so that when X is 1-random, X has degree $0'$. The last corollary follows from Franklin's result that there is a Π^0_1 class of Schnorr trivial reals [5].

2.4 Relatively Simple and Above

We show that every 1-generic real X is relatively simple and above by an argument similar to the arithmetic form of the proof of Theorem 2.1.1. To obtain a witness Y showing that X is relatively simple and above, we will need to find a sufficiently generic order in which to enumerate the elements of X. We find that X itself can be used to compute this order.

Lemma 2.4.1. *Let X be 1-generic. Then X is relatively simple and above.*

Proof. Let $Y = \{\langle n,m\rangle \mid n \in X \ \wedge \ \langle n,m\rangle \notin X\}$. Then $Y \leq_T X$ and X is r.e. (Y) since $n \in X$ if and only if $\exists m\,[\langle n,m\rangle \in Y]$ (since X is generic, we can't have $\langle n,m\rangle \in X$ for every m). It remains to show that there is no infinite $Z \subseteq \overline{X}$ such that Z is r.e. (Y).

Suppose towards a contradiction there is an infinite $Z \subseteq \overline{X}$ such that $Z = W_k^Y$ for some k. We define a function $j : 2^{<\omega} \to 2^{<\omega}$ such that $Y = j(X)$. Let $j(\sigma(\langle n, m\rangle)) = 1$ iff $\sigma(n) = 1$ and $\sigma(\langle n, m\rangle) = 0$. Since $X \cap W_k^{j(X)} = \emptyset$ and X is 1-generic, we can find a condition such that for every extension τ we have $\tau \cap W_k^{j(\tau)} = \emptyset$. We will then get a contradiction by adding an element to τ without changing $j(\tau)$.

Let $S = \{\sigma \mid \exists n\, [n \in \sigma \;\wedge\; n \in W_k^{j(\sigma)}]\}$. Then $X \notin S$, so let l be such that for every τ extending $X|l$ we have $\tau \notin S$. Since Z is infinite, let $p \in Z$ with $p > l$ and let $t > l$ be such that $p \in W_k^{j(X|t)}$. We note that for any $\sigma \supseteq X|l$ such that $j(\sigma) = j(X|t)$ we have $p \in W_k^{j(\sigma)}$ and $\sigma \notin S$, so $p \notin \sigma$. We can now obtain a contradiction.

Claim. *There is a $\sigma \supseteq X|l$ such that $j(\sigma) = j(X|t)$ and $p \in \sigma$.*

Proof. We define a sequence of strings σ_i of length t inductively. Let $\sigma_0 = X|t$ and $\sigma_1 = \sigma_0$ except $\sigma_1(p) = 1$. At each stage we will remove all witnesses of changes made in the previous stage. We assume our pairing function is such that $\langle m, n\rangle > \max(m, n)$ for all m, n.

For stage $i \geq 2$ we let $\sigma_i(\langle b, a\rangle) = 1$ if $\sigma_{i-1}(b) \neq \sigma_{i-2}(b)$ and let $\sigma_i(\langle b, a\rangle) = \sigma_{i-1}(\langle b, a\rangle)$ otherwise. We note that since $\langle b, a\rangle > b$, the least m such that $\sigma_i(m) \neq \sigma_{i-1}(m)$ is strictly increasing by stage. Hence for some stage we have $\sigma_i = \sigma_{i-1}$, and we let σ be this σ_i.

We have $p \in \sigma$ and note $\sigma \supseteq X|l$ since $p > l$. It remains to show that $j(\sigma) = j(X|t)$. Let n, m be arbitrary such that $[j(X|t)](\langle n, m\rangle) = 1$. Then $X(n) = 1$ and $X(\langle n, m\rangle) = 0$. So at every stage i, $\sigma_i(n) = 1$ and $\sigma_i(\langle n, m\rangle) = \sigma_{i-1}(\langle n, m\rangle)$. Hence $\sigma(n) = 1$ and $\sigma(\langle n, m\rangle) = 0$ so $[j(\sigma)](\langle n, m\rangle) = 1$. Conversely, let n, m be arbitrary such that $[j(\sigma)](\langle n, m\rangle) = 1$. Then $\sigma(n) = 1$ and $\sigma(\langle n, m\rangle) = 0$. The later implies $X(\langle n, m\rangle) = 0$. Suppose $X(n) = 0$. Let i

be least such that $\sigma_i(n) = 1$. Then $\sigma_{i+1}(\langle n, m\rangle) = 1$, so $\sigma(\langle n, m\rangle) = 1$ for a contradiction. Hence $X(n) = 1$ so $[j(X|t)](\langle n, m\rangle) = 1$. Therefore $j(\sigma) = j(X|t)$. □

Thus, Z is not r.e. (Y). Therefore X is relatively simple and above. □

2.5 Other Results

We saw in Remark 2.1.2 that if X is relatively r.e. then we can find a witness Y which is close to X. On the other hand, we can also choose a witness Y which is not even arithmetic in X. To do this, we need to use a modified version of Theorem 2.1.1.

Corollary 2.5.1. *Let X and T be reals such that $\overline{X} \not\leq_e X \oplus T \oplus \overline{T}$. Then there is a $Y \geq_T T$ such that X is r.e. (Y) and $X \not\leq_T Y$.*

Proof. The proof is very similar to that of theorem 2.1.1. We use the same partial order and let G be generic in $X \oplus T$. We then let $Y = m(G \oplus T)$ and relativize the verification to T. At the end we use the fact that we have a $\Sigma_1(T)$ set witnessing $\overline{X} \leq_{e_1(T)} X$ to get a Σ_1 set witnessing $\overline{X} \leq_{e_1} X \oplus T \oplus \overline{T}$. □

Let X be relatively r.e. and let G be an arithmetic generic (X). We can now obtain a witness Y which is not arithmetic in X by choosing Y so that $G \leq_T Y$.

Lemma 2.5.2. *Let X be a real such that $\overline{X} \not\leq_e X$. Then there is a real Y which is not arithmetic in X such that X is r.e. (Y) and $X \not\leq_T Y$.*

Proof. Let G be an arithmetic generic (X). By the above lemma, it suffices to show that $\overline{X} \not\leq_e X \oplus G \oplus \overline{G}$. Suppose not, witnessed by C. Let S be given by

$$S = \{\sigma \in 2^{<\omega} \mid \exists n \in X \, \exists F \subset (3 \cdot \text{length}(\sigma))[\langle n, F\rangle \in C \ \wedge \ F \subseteq X|\text{length}(\sigma) \oplus \sigma \oplus \overline{\sigma}]\}$$

Then $G \notin S$ so let l be such that for all $\tau \supseteq G|l$ we have $\tau \notin S$. Let

$$B = \{\langle n, A\rangle \mid \exists \sigma \supseteq G|l \, \exists F \, [\langle n, F\rangle \in C \ \wedge \ F \subseteq A \oplus \sigma \oplus \overline{\sigma}]\}$$

Then B witnesses that $\overline{X} \leq_e X$ for a contradiction. $\square$

Chapter 3

Reals n-generic relative to some perfect tree

3.1 Co-Countably Many Reals

We wish to show that the set of reals not n-generic relative to any perfect tree is countable. D. Martin [14] used Borel determinacy to show that any property which is Borel and cofinal in the Turing degrees is represented on every degree in a cone of Turing degrees. The base of this cone is the complexity of the winning strategy for an associated game.

Theorem 3.1.1 (Martin [14]). *Let $\mathcal{B}$ be a Borel set of reals such that for every Turing degree $\mathbf{d}$ there is an $\mathbf{e} \geq_T \mathbf{d}$ and an X in $\mathbf{e}$ such that $X \in \mathcal{B}$. Then there is a degree $\mathbf{c}$ such that for all $\mathbf{b} \geq_T \mathbf{c}$ there is a Y in $\mathbf{b}$ such that $Y \in \mathcal{B}$.*

Proof. Consider a two person game where player I constructs a real X and player II constructs a real Y. Play alternates between the players, each adding the next digit to the real they are constructing for their turn. Player I wins iff $Y \leq_T X$ and either $X \not\leq_T Y$ or $X \in \mathcal{B}$.

By Borel determinacy, there exists a winning strategy σ. Suppose σ is a winning strategy for II. Since $\mathcal{B}$ is cofinal in the Turing degrees, let $Z \in \mathcal{B}$ with $\sigma \leq_T Z$. Let I play Z and II play according to σ resulting in Y. But then since $\sigma \leq_T Z$, we have $Y \leq_T Z$ with $Z \in \mathcal{B}$ so I wins for a contradiction. Hence σ must be a winning strategy for I.

Let $Z \geq_T \sigma$ be an arbitrary real in the cone above σ. Have II play Z and I play according to σ, resulting in X. Since $\sigma \leq_T Z$ we have $X \leq_T Z$. Since I wins, $Z \leq_T X$ and $X \in \mathcal{B}$. Hence for any Z in the cone, there is an $X \in \mathcal{B}$ such that $X \equiv_T Z$. □

Reimann and Slaman [17] have developed a powerful way to relativize this lemma. Let $\mathcal{B} \subseteq 2^\omega \times 2^\omega$ denote a set of reals where the first real holds some property relative to the second. Let $\mathcal{B}^Z = \{X \mid (X, Z) \in \mathcal{B}\}$ and let the notation $X \equiv_{T,A} Y$ mean $X \oplus A \equiv_T Y \oplus A$. Suppose that for every Z the set $\mathcal{B}^Z$ is Borel in Z and cofinal in the Turing degrees as in the above method for generating a cone. They prove that for all but countably many reals X, there exist reals Y and G such that $X \equiv_{T,G} Y$ and $Y \in \mathcal{B}^G$.

Let β be the least ordinal such that L_β satisfies (enough) ZFC and let $X \notin L_\beta$ be arbitrary (L_β is countable). Reimann and Slaman use Kumabe-Slaman forcing to find a real G such that $L_\beta[G] \models$ ZFC and every element of $2^\omega \cap L_\beta[G]$ is recursive in $X \oplus G$. In particular, the strategy for the associated game relative to G is recursive in $X \oplus G$. So by Lemma 3.1.1 relativized to G there exists $Y \in \mathcal{B}^G$ with $Y \equiv_{T,G} X$.

Thus to prove all but countably many reals are n-generic relative to some perfect tree, we need to find a set $\mathcal{B}$ such that for any X, Y, G with $Y \equiv_{T,G} X$ and $Y \in \mathcal{B}^G$ we have

X n-generic relative to some perfect tree. $\mathcal{B}$ must also be Borel and such that for every Z the set $\mathcal{B}^Z$ is cofinal in the Turing degrees. We find it suffices to let $\mathcal{B}$ be the set of reals of Turing degree $X \oplus A$ for any X, A such that X is $(n+1)$-generic (A). We use the following lemma.

Lemma 3.1.2. *Let $n \geq 2$, A be a set, X be n-generic (A), and $X \equiv_{T,A} Y$. Then Y is $(n-1)$-generic relative to some perfect tree.*

Proof. Let $\Psi : X \to Y$ and $\Phi : Y \to X$ be A-recursive Turing reductions that witness $X \equiv_{T,A} Y$. Since X is at least 2-generic (A), let $p \in X$ be such that $p \Vdash \Phi \circ \Psi = \mathrm{id} \ \wedge \ \Psi$ total. Let $T = \{\sigma \mid \exists q \supseteq p[\sigma \subseteq \Psi(q)]\}$. T is a perfect tree by our choice of p. We claim that Y is $(n-1)$-generic relative to T.

Let S be an arbitrary $\Sigma^0_{n-1}(T)$ set. We consider the pullback $\Psi^{-1}(S) = \{x \mid \exists y\,[\Psi(x) \supseteq y \ \wedge \ y \in S]\}$. T is $\Sigma^0_1(A)$ so S is $\Sigma^0_n(A)$ and $\Psi^{-1}(S)$ is $\Sigma^0_n(A)$. We now apply the genericity of X for the pullback to get the genericity of Y for S.

Since X is n-generic (A) we have two possible cases.

Case 1: $\exists n[X|n \in \Psi^{-1}(S)]$. We then let m be such that $Y|m \subseteq \Psi(X|n)$ and $Y|m \in S$.

Case 2: $\exists n \forall q \supseteq X|n[q \notin \Psi^{-1}(S)]$. Let m be such that $\Phi(Y|m) \supseteq X|n$. We will show $Y|m$ witnesses Y is $(n-1)$-generic relative to T for S. Consider an arbitrary $r \in T$ such that $r \supseteq Y|m$. Since $r \in T$, let q be such that $\Psi(q) \supseteq r$ and $q \supseteq p$. We note $q \supseteq \Phi(\Psi(q)) \supseteq \Phi(r) \supseteq \Phi(Y|m) \supseteq X|n$. Hence by the condition for this case, $q \notin \Psi^{-1}(S)$ so $r \notin S$. Since r is arbitrary, for all $r \supseteq Y|m$ with $r \in T$ we have $r \notin S$. □

We note that a similar proof can be used to show for $n \geq 1$ that sets in the same truth table degree as an n-generic are n-generic relative to some perfect tree.

We can now use the approach outlined above.

Theorem 3.1.3. *For every $n \in \omega$, the set of reals not n-generic relative to some perfect tree is countable.*

Proof. Fix $n \in \omega$ and let

$$\mathcal{B} = \{(x, g) \mid \exists a \exists h[h \text{ is } (n+1)\text{-generic } (a \oplus g) \text{ and } x \equiv_{T,g} a \oplus h\}$$

$\mathcal{B}$ is arithmetic (since $a, h \leq_T x \oplus g$) so $\mathcal{B}$ is Borel. Given any reals C and Z, we let H be $(n+1)$-generic $(C \oplus Z)$ and $X = H \oplus C$ to get $X \in \mathcal{B}^Z$ with $X \geq_T C$. Hence $\mathcal{B}^Z$ is cofinal in the Turing degrees. By the theorem of Reimann and Slaman [17] noted above, for all but countably many reals X, there exist Y and G such that $X \equiv_{T,G} Y$ and $Y \in \mathcal{B}^G$.

Thus there exist reals A and H such that $A \oplus H \equiv_{T,G} Y$ and H is $(n+1)$-generic $(A \oplus G)$. Hence $Y \equiv_{T,A\oplus G} H$, so $X \equiv_{T,A\oplus G} H$. By Lemma 3.1.2, X is n-generic relative to some perfect tree. Therefore, all but countably many reals are n-generic relative to some perfect tree. □

3.2 ZFC$^-$ Required

If we examine the proof that the set of reals not n-generic relative to any perfect tree is countable, we see that the greatest use of the axioms of ZFC comes from the application of Borel determinacy. The proof uses determinacy of a Π^0_{n+3} game on ω^ω, so it requires ZFC$^-$ and the existence of n iterates of the power set of ω. We prove that for sufficiently large n this is essentially the best possible result. In particular, we prove that for any finite k the statement "For all n, the set of reals not n-generic relative to any perfect tree is countable" cannot be proved from ZFC^- and k iterates of the power set of ω. This suggests the set of reals not n-generic relative to any perfect tree is a countable set of considerable size and complexity.

Theorem 3.2.1. *For every $k \in \omega$ the statement "For all n, the set of reals not n-generic relative to any perfect tree is countable" cannot be proved from ZFC$^-$+ "$\exists k$ iterates of the power set of ω".*

To prove this theorem we use a template developed by Reimann and Slaman [17] for reals random relative to a continuous measure. We work with the case $k = 0$; the general case follows the same pattern. Let λ be the least ordinal such that $L_\lambda \models \mathrm{ZFC}^-$ and let O be the set of limit ordinals below λ. Let M_α, for $\alpha \in O$, denote master codes. These are the elementary diagrams of canonical countings of L_α. Reimann and Slaman prove the theorem by showing that for some fixed n, for every $\alpha \in O$, the master code M_α is not n-random relative to a continuous measure.

To show this, they assume towards a contradiction that some M_β is n-random relative to the measure μ. It is arithmetic to say that M is a master code for an ω-model of "$V = L_\alpha$ and α a limit and $\alpha \not\geq \lambda$". Note such an ω-model need not be well-founded. They show it is also arithmetic to require that for all such M and N and some fixed $m \in \omega$ either one coded model embeds into the other or there is a $\Sigma^0_m(M \oplus N)$ set witnessing the ill-foundedness of one of the coded models.

Reimann and Slaman define a set $\mathcal{M}$, arithmetic in μ, of such psuedo-master codes which are recursive in μ and not shown to be ill-founded by such a comparison. They then define an order on $\mathcal{M}$ such that the well founded part of this order, I, is arithmetic in

$\mu \oplus M_\beta$ and equals the set of $M \in \mathcal{M}$ which are actual master codes M_α. Since random sets cannot accelerate the calculation of well-foundedness, I is arithmetic in μ.

Let $\gamma \leq \beta$ be least such that $M_\gamma \not\leq_T \mu$. Since $\gamma < \lambda$ there is a real $X \in \mathrm{Def}(L_\gamma)\backslash L_\gamma$. By taking a Skolem hull of the parameters defining X, Reimann and Slaman show that M_γ is arithmetic in I, hence arithmetic in μ, and $M_\gamma \leq_T M_\beta$. Since randomness cannot accelerate arithmetic definability, $M_\gamma \leq_T \mu$ for a contradiction.

This proof uses only two facts about randomness. Namely, that it cannot accelerate arithmetic definability or calculations of well-foundedness. To complete a similar proof for genericity with the perfect tree T in place of the measure μ we need only demonstrate the corresponding facts for genericity. For n sufficiently large relative to a fixed k, m and G n-generic relative to the perfect tree T, we must show:

1. If A is $\Sigma^0_k(T)$ and $\Sigma^0_m(G)$ then A is $\Sigma^0_m(T)$.
2. If WF is the well founded part of a linear order recursive in T and $WF \leq_T G \oplus T$ then $WF \leq_T T$.

We can routinely relativize to a perfect tree the proof that for reals A, G where G is k-generic and A is Σ^0_k and $\Sigma^0_m(G)$, we get that A is Σ^0_m. To prove the second fact, we use the following lemma.

Lemma 3.2.2. *Let T be a perfect tree and L a code for a linear order where $L \leq_T T$. Let WF be the code for the well founded part of L. Let G be 2-generic relative to T and such that $WF \leq_T G \oplus T$. Then $WF \leq_T T$.*

Proof. By the first fact above, it suffices to show WF is $\Sigma^0_j(T)$ for some j. Let Φ be a T-recursive Turing reduction such that $\Phi(G) = WF$. For $b \in L$ let $P(b)$ be the set of reals which code initial segments of L below b. Let $R(b) \leq_T L$ be the tree defined below such that $P(b)$ is the set of paths through $R(b)$.

$$R(b) = \{\sigma \mid \forall m, n < \text{length}(\sigma)[(m \in \sigma \to m \leq_L b) \ \wedge$$
$$((m \in \sigma \ \wedge \ n \leq_L m) \to n \in \sigma)]\}$$

We define Q as the set of strings in T below which Φ does not split on T.

$$Q = \{\sigma \in T \mid \neg\exists\tau, \gamma \in T[\tau, \gamma \supseteq \sigma \ \wedge \ \Phi(\tau) \perp \Phi(\gamma)]\}$$

Suppose for some n, $G|n \in Q$. Then we can calculate $WF(m)$, i.e. whether $m \in WF$, by looking for the first $\sigma \in T$ such that $\sigma \supseteq G|n$ and $[\Phi(\sigma)](m)\downarrow$ and taking its value. Hence $WF \leq_T T$ and we are done. Thus we may assume for all n, $G|n \notin Q$. Since G is 2-generic relative to T, there is an l such that for all $\tau \supseteq G|l$ with $\tau \in T$ we have $\tau \notin Q$.

We will use the fact that $b \in WF$ iff $WF \notin P(b)$. Let $S = \{\sigma \mid \Phi(\sigma) \notin R(b)\}$. We will determine if $b \in WF$ by checking for the existence of a real which is generic for S and computes an element of $P(b)$ using Φ. Let $\Theta(b)$ be the statement

$$\exists\sigma \in T[\sigma \supseteq G|l \ \wedge \ \forall\tau \in T[\tau \supseteq \sigma \to \Phi(\tau) \in R(b)]]$$

Claim. *$b \notin WF \Leftrightarrow \Theta(b)$.*

Proof. ($\Longrightarrow$) $WF \in P(b)$ since $b \notin WF$. Hence $\Phi(G) \in P(b)$ so $G \notin S$. Since G is 1-generic relative to T, there is a k such that for all $\tau \in T$ with $\tau \supseteq G|k$ we have $\tau \notin S$. Thus $G|k$ witnesses $\Theta(b)$.

($\Longleftarrow$) Let σ witness $\Theta(b)$. Then for all $\tau \in T$ with $\tau \supseteq \sigma$ we have $\Phi(\tau) \in R(b)$. Also, since $\sigma \supseteq G|l$, for all such τ we have $\tau \notin Q$ so Φ splits on T below τ. Using these facts we can construct a perfect subtree of $R(b)$ by applying Φ to T below σ. Hence $P(b)$ is uncountable. Since there are only countably many well founded initial segments of L, $b \notin WF$. $\square$

By the claim WF is $\Sigma^0_2(T)$ as desired. Hence $WF \leq_T T$. $\square$

3.3 Iterated Hyperjumps

We now look at the set of reals which are n-generic relative to some perfect tree for lower values of n. We still find that the set of reals not n-generic relative to any perfect tree is a large countable set. It contains reals of high complexity and its countability cannot be proved in large fragments of second order arithmetic. We show that the finite iterates of the hyperjump, $\mathcal{O}^{(n)}$, are not 2-generic relative to any perfect tree and the iterates $\mathcal{O}^{(\alpha)}$ are not 5-generic relative to any perfect tree for any α below the least λ such that $\sup_{\beta<\lambda}(\beta$th admissible$) = \lambda$.

For simplicity, we start with the case of $\mathcal{O}$. This set can be viewed as $\{e \mid U_e \text{ is well-founded}\}$ where U_e denotes the eth recursive tree in $\omega^{<\omega}$. We note $\mathcal{O}$ then has the property that the well-foundedness of subtrees cannot contradict the decision made for the parent tree. This can be characterized by a Σ^0_2 set, S, so that if $\mathcal{O}$ were 2-generic relative to some T then T would be able to calculate $\mathcal{O}$ by tracing subtrees.

Lemma 3.3.1. *$\mathcal{O}$ is not 2-generic relative to any perfect tree.*

Proof. Suppose not, witnessed by T. Define $\mathcal{O} = \{e \mid U_e \text{ is well-founded}\}$ as above. Let h

be a recursive function defined by $U_{h(e,\gamma)} = \{\sigma \in U_e \mid \sigma \subseteq \gamma \ \vee \ \sigma \supseteq \gamma\}$. Let

$$S = \{\tau \in T \mid \exists n \exists l[\tau(n) = 0 \ \wedge \ \neg(\exists\gamma \in U_n)(\exists\theta \in T)[\text{length}(\gamma) \geq l \ \wedge \\ \theta \supseteq \tau \ \wedge \ \theta(h(n,\gamma)) = 0]]\}$$

The set S contains finite strings τ which say some tree U_n is ill-founded, but for some length l, there is no extension of τ in T that says some subtree of U_n with root length at least l is ill-founded. In short, τ says U_n is ill-founded but there is no sequence of extensions in T to witness it.

$\mathcal{O}$ has a sequence of subtrees witnessing the ill-foundedness of a tree, found simply by descending along any infinite path. Hence, $\mathcal{O} \notin S$. Since S is $\Sigma^0_2(T)$ and we have assumed $\mathcal{O}$ is 2-generic relative to T, we let k be such that for any $\sigma \in T$ extending $\mathcal{O}|k$ we have $\sigma \notin S$. We can now use the fact that these extensions are sufficiently well behaved to calculate $\mathcal{O}$ from T.

Claim. *For any number e, we have $e \in \mathcal{O} \iff \neg\exists\sigma \in T[\sigma \supseteq \mathcal{O}|k \ \wedge \ \sigma(e) = 0]$.*

Proof. ($\Longleftarrow$) Let $\sigma = \mathcal{O}|\max(k,e)+1$. $\sigma \in T$ since $\mathcal{O}$ is a path in T, so $\sigma(e) = \mathcal{O}(e) = 1$. Hence $e \in \mathcal{O}$.

($\Longrightarrow$) Let $e \in \mathcal{O}$ and suppose the conclusion fails, witnessed by σ. U_e is well-founded since $e \in \mathcal{O}$. We will construct an infinite path through U_e to get the desired contradiction. We use an induction to simultaneously construct paths γ through U_e and θ through T. Let j_0 denote e and j_{m+1} denote $h(j_m, \gamma_{m+1})$. We maintain inductively that $\theta_m(j_m) = 0$.

We begin with $\gamma_0 = \langle\rangle$ and $\theta_0 = \sigma$ and note $\theta_0(j_0)) = \sigma(e) = 0$ by our assumption.

Let γ_m and θ_m be given. $\theta_m \supseteq \sigma \supseteq \mathcal{O}|k$ so $\theta_m \notin S$. Hence we have

$$\forall n \forall l[\theta_m(n) \neq 0 \ \vee \ \exists \alpha \in U_n \exists \beta \in T[\text{length}(\alpha) \geq l \ \wedge \ \beta \supseteq \theta_m \ \wedge$$
$$\beta(h(n, \alpha)) = 0]]$$

Choosing $n = j_m$ and l =length(γ_m)+1 and noting by our induction hypothesis $\theta_m(j_m) = 0$, we get

$$\exists \alpha \in U_{j_m} \exists \beta \in T[\text{length}(\alpha) \geq \text{length}(\gamma_m) + 1 \ \wedge \ \beta \supseteq \theta_m \ \wedge$$
$$\beta(h(j_m, \alpha)) = 0]$$

We now let $\gamma_{m+1} = \alpha$ and $\theta_{m+1} = \beta$. We note that $\gamma_{m+1} \supseteq \gamma_m$ since $\gamma_{m+1} \in U_{h(j_{m-1}, \gamma_m)}$ and that $\theta_{m+1}(j_{m+1}) = \theta_{m+1}(h(j_m, \gamma_{m+1})) = 0$, completing the induction. □

Thus $\mathcal{O}$ is $\Pi^0_1(T)$, contradicting $\mathcal{O}$ being 2-generic relative to T. □

For the successor case, we apply the same ideas used in the above lemma to the column of $\mathcal{O}^{(n)}$ which computes $\mathcal{O}$.

Lemma 3.3.2. *Let $X \geq_T \mathcal{O}$ be 2-generic relative to the perfect tree T. Then $T \geq_T \mathcal{O}$.*

Proof. Let Φ be a Turing reduction such that $\Phi(X) = \mathcal{O}$. We define S as before, this time for the image under Φ.

$$S = \{\tau \in T \mid \exists n \exists l[[\Phi(\tau)](n) = 0 \ \wedge \ \neg(\exists \gamma \in U_n)(\exists \theta \in T)[\text{length}(\gamma) \geq l \ \wedge$$
$$\theta \supseteq \tau \ \wedge \ [\Phi(\theta)](h(n, \gamma)) = 0]]\}$$

We note S is $\Sigma^0_2(T)$ and $X \notin S$. Since X is 2-generic relative to T, we let k be such that for any $\sigma \in T$ extending $X|k$ we have $\sigma \notin S$. We now claim that for any e, we have $e \in \mathcal{O}$

if and only if there does not exist a $\sigma \in T$ with $\sigma \supseteq X|k$ and $[\Phi(\sigma)](e) = 0$. This is proved in substantially the same manner as the claim in the previous lemma. As a result, $\mathcal{O}$ is $\Pi^0_1(T)$. Since X is 2-generic relative to T and $\mathcal{O} \leq_T X$, we get $\mathcal{O} \leq_T T$ as desired. □

Corollary 3.3.3. *For all $n \in \omega$, $\mathcal{O}^{(n)}$ is not 2-generic relative to any perfect tree.*

Proof. Fix n and suppose not, witnessed by T. We show by induction on $m \leq n$ that $\mathcal{O}^{(m)} \leq_T T$. Given $\mathcal{O}^{(m)} \leq_T T$, we relativize Lemma 3.3.2 to $\mathcal{O}^{(m)}$ to get $\mathcal{O}^{(m+1)} \leq_T T$, completing the induction. Hence $\mathcal{O}^{(n)} \leq_T T$, contradicting our assumption that $\mathcal{O}^{(n)}$ is 2-generic relative to T. □

Corollary 3.3.4. *The statement "All but countably many reals are 2-generic relative to some perfect tree" fails to hold in Π^1_1-CA.*

Proof. Consider the standard model of Π^1_1-CA containing the reals X such that $\exists n[X \leq_T \mathcal{O}^{(n)}]$. The set $\{\mathcal{O}^{(n)} \mid n \in \omega\}$ fails to be countable in this model. □

To handle limit ordinals, we use a lemma in the style of Enderton and Putnam [4].

Lemma 3.3.5 (Slaman [20]). *Let A be a set and λ a recursive limit ordinal. Suppose that for all $\beta < \lambda$, $\mathcal{O}^{(\beta)} \leq_T A$. Then $\mathcal{O}^{(\lambda)}$ is $\Sigma^0_5(A)$.*

Proof. We continue to use $\mathcal{O} = \{e \mid U_e \text{ is well founded}\}$ where U_e denotes the eth recursive tree in $\omega^{<\omega}$. Since $\mathcal{O} \leq_T A$ we can define $\mathcal{O}$ from A by noting that U_e is well founded iff U_e has no infinite path recursive in A. Hence $\mathcal{O}$ is uniformly $\Pi^0_3(A)$. Similarly, we can get $\mathcal{O}^{\mathcal{O}}$ is uniformly $\Pi^0_4(A)$ by $X = \mathcal{O} \oplus \mathcal{O}^{\mathcal{O}}$ iff

$$(X)_0 = \mathcal{O} \ \wedge \ (e \in (X)_1 \leftrightarrow U_e^{(X)_0} \text{ has no infinite path recursive in } A)$$

where $(X)_0$ and $(X)_1$ denote the two columns of X.

We extend this idea to find a uniform definition for $\mathcal{O}^{(\lambda)}$. Fix a system of notations, o, for λ. We have

$$(b,k) \in \mathcal{O}^{(\lambda)} \Leftrightarrow \exists m[\{m\}^A = Y \ \wedge \ k \in (Y)_b \ \wedge \ [\forall c \forall d[o(c) < o(b) \to$$

$$((o(c) = o(d) + 1 \to \Gamma((Y)_d, (Y)_c)) \wedge$$

$$(o(c) \text{ a limit ordinal } \to \forall n \forall p[(Y)_{c_n}(p) = ((Y)_c)_n(p)]))]]]$$

where $c_0, c_1, c_2, \ldots$ is the fundamental sequence for $o(c)$ and $\Gamma(X,Z)$ is the statement

$$\forall e[e \in Z \leftrightarrow U_e^X \text{ has no infinite path recursive in } A]$$

Then Γ is $\Pi^0_4(A)$ so $\mathcal{O}^{(\lambda)}$ is $\Sigma^0_5(A)$. □

If we repeat the proof with $\overline{\mathcal{O}^{(\lambda)}}$ we improve the result slightly to $\mathcal{O}^{(\lambda)}$ is $\Delta^0_5(A)$.

Now we can complete our induction.

Theorem 3.3.6. *Let λ be the least ordinal such that $\sup_{\beta<\lambda}(\beta th \text{ admissible}) = \lambda$. Then for all $\alpha < \lambda$ we have $\mathcal{O}^{(\alpha)}$ is not 5-generic relative to any perfect tree.*

Proof. Suppose not, witnessed by β and T. We define the function f by $f(0) = \omega_1^{CK}$, $f(\delta+1) =$ least admissible greater than $f(\delta)$, and for limit δ, $f(\delta) = \sup_{\xi<\delta} f(\xi)$. We note that λ is the least fixed point of f. Using the fact that $\omega_1^{\mathcal{O}^{(\delta)}} < \omega_1^{\mathcal{O}^{(\delta+1)}}$ for any δ [18], we see by induction that $f(\delta) < \omega_1^{\mathcal{O}^{(\delta)}}$ for all δ.

Let α be least such that $\mathcal{O}^{(\alpha)} \not\leq_T T$. Then $\alpha \leq \beta < \lambda$ so $\alpha < f(\alpha)$. By Lemma 3.3.2, α is a nonzero limit ordinal so choose $\gamma < \alpha$ such that $\alpha < f(\gamma)$. Then $\alpha < \omega_1^{\mathcal{O}^{(\gamma)}}$ so we can fix a system of notations, o, for α recursive in $\mathcal{O}^{(\gamma+1)}$. Since $\gamma + 1 < \alpha$ we have

$\mathcal{O}^{(\gamma+1)} \leq_T T$. We now apply Lemma 3.3.5 using o and T to get that $\mathcal{O}^{(\alpha)} \leq_T T$ for a contradiction. □

3.4 1-generics

In the 1-generic case, we can use a variety of approaches to identify sets of reals that are 1-generic relative to some perfect tree and sets whose members cannot have this property.

A real is said to be ranked if it is a member of a countable Π^0_1 set. Equivalently, a real is ranked if it is a path through a recursive tree with no perfect subtrees. The reader is referred to Cenzer et al. [3] for details on the topic, including a proof that for all recursive ordinals α there is a ranked set of degree $\mathbf{0}^{(\alpha)}$. Here we demonstrate these reals are not 1-generic relative to any perfect tree.

Lemma 3.4.1. *If X is 1-generic relative to some perfect tree, then X is not ranked.*

Proof. Suppose not. Let X be 1-generic relative to the perfect tree T and a path through the recursive tree U with no perfect subtrees. Let $S = \{\sigma \in T \mid \sigma \notin U\}$. Then S is recursive in T and $X \notin S$, so there exists an n such that no $\tau \in T$ extending $X|n$ is in S. Hence for every $\tau \in T$ such that $\tau \supseteq X|n$ we have $\tau \in U$. But then U has a perfect subtree, for a contradiction. □

In fact, the above proof shows reals recursively generic relative to some perfect tree are not ranked. It follows from Cenzer et al. [3] and this lemma that there are reals arbitrarily high in the hyperarithmetic degrees which are not 1-generic relative to any perfect tree. We note the proof of Lemma 3.1.2 can be relativized to an initial perfect tree. Using this we observe that no Turing degree can contain both a ranked set and a real 2-generic relative to some perfect tree (and no truth table degree a ranked set and a real 1-generic relative to some perfect tree). Hence no Δ^0_2 set is 2-generic relative to some perfect tree, and the degrees $\mathbf{0}^{(\alpha)}$ for any recursive α contain no reals 2-generic relative to some perfect tree.

The techniques used in Lemmas 3.2.2 and 3.4.1 can be used to further restrict possibilities for ranked reals.

Theorem 3.4.2. *Let G be a 2-generic real and let X be a nonrecursive real with $X \leq_T G$. Then X is not ranked.*

Proof. Suppose not. Let U be a recursive tree with no perfect subtrees such that X is a path through U. Let Φ be a Turing reduction such that $\Phi(G) = X$. Let Q be the set of strings below which Φ does not split on U.

$$Q = \{\sigma \in 2^{<\omega} \mid \neg\exists\tau, \gamma \supseteq \sigma\, [\Phi(\tau), \Phi(\gamma) \in U \ \wedge\ \Phi(\tau) \perp \Phi(\gamma)\}$$

Suppose for some l we have $G|l \in Q$. Then we could calculate $X(n)$ by searching for the first $\sigma \supseteq G|l$ such that $\Phi(\sigma) \in U$ and has length at least n. We then let $X(n) = [\Phi(\sigma)](n)$. Hence X would be recursive for a contradiction. Thus $G \notin Q$ and we let l be such that for all $\tau \supseteq G|l$ we have $\tau \notin Q$.

Let $S = \{\sigma \supseteq G|l \mid \Phi(\sigma) \notin U\}$. Then $G \notin S$ so let $m > l$ be such that for all $\tau \supseteq G|m$ we have $\tau \notin S$. Hence for all $\sigma \supseteq G|m$ we have $\Phi(\sigma) \in U$ and Φ splits below σ. Thus we can build a perfect subtree of U by applying Φ below $G|m$, for a contradiction. ☐

We note this proof shows the statement holds for Π_1-generics.

We can also attempt to classify which reals are 1-generic relative to some perfect tree by use of the r.e. (Ershov) and REA hierarchies. The reader is referred to Jockusch and Shore [8] for details on these hierarchies. We begin by observing that no real whose degree is at a finite level of the REA hierarchy (hence also the r.e. hierarchy) is 1-generic relative to some perfect tree.

Lemma 3.4.3 (Slaman [19])**.** *Let $n \in \omega$, X a real of n-REA degree. Then X is not 1-generic relative to any perfect tree.*

Proof. Fix n and X and let W be an n-REA set with $X \equiv_T W$. Let $W_1, W_2, \ldots$ $W_n = W$ witness that W is n-REA; for all $i \leq n$ we have $W_i \leq_T W_{i+1}$ and W_{i+1} is r.e.(W_i). Suppose X is 1-generic relative to T. We show by induction that for all $m \leq n$ we have $W_m \leq_T T$ using the following claim:

Claim. *Let Y be r.e.(T) and $Y \leq_T X$. Then $Y \leq_T T$.*

Proof. It suffices to show $\overline{Y}$ is r.e.(T). Since $Y \leq_T X$, let $\{e\}^X = Y$. Let

$$S = \{q \mid \exists n[\{e\}^q(n)\downarrow = 0 \wedge n \in Y]\}$$

We note $X \notin S$ and S is r.e.(T) since Y is r.e.(T). Hence for some l, every q extending $X|l$ is not in S. We can now describe $\overline{Y}$ by noting that $n \in \overline{Y}$ iff

$\exists q \supseteq X|l[\{e\}^q(n)\downarrow = 0]$. Hence $\overline{Y}$ is r.e.(T) as desired. □

For the induction, given $W_m \leq_T T$ we note that W_{m+1} is r.e.(W_m), hence r.e.(T), and apply the claim to W_{m+1}. As a result $W \leq_T T$ so $X \leq_T T$ for a contradiction. □

We might next hope to show sets of ω-REA degree are not 1-generic relative to any perfect tree. However, we cannot even do this for sets which are ω-r.e. In proving the Friedberg Inversion Theorem for the truth table degrees, J. Mohrherr [16] showed by a reduction that there is a 1-generic G such that $G \leq_{tt} 0'$, hence G is ω-r.e. Here we provide a direct construction. We use the definition that X is ω-r.e. if for some partial recursive $\psi : \omega \times \omega \to 2$ we have $X(n) = \psi(b, n)$ where b is least such that $\psi(b, n)\downarrow$.

Lemma 3.4.4. *There is a 1-generic real which is ω-r.e.*

Proof. In this construction we extend to meet the first r.e. set we find while still looking for earlier r.e. sets skipped over. If we find a set that has been skipped, we start over again from that point. We start with a preliminary construction where we don't restart in order to find a recursive bound, f, on the number of changes we may need to make.

We recursively construct $f : \omega \to \omega$ and $\sigma \in 2^{\omega}$ in stages. Let $f(0) = 0$ and $\sigma_0 = \langle\rangle$. At stage $n+1$ we search simultaneously for $e > f(\text{length}(\sigma_n)), \tau \supseteq \sigma_n$, and s to find $\{e\}^{\tau}_{s}\downarrow$. We then let $\sigma_{n+1} = \tau$ and extend f by setting $f(k) = e$ for all k such that $\text{length}(\sigma_n) < k \leq \text{length}(\sigma_{n+1})$.

For the main construction, we build in stages our generic $X \in 2^{\omega}$ and the partial recursive witness, ψ, that X is ω-r.e. We also use some numeric variables for bookkeeping. r denotes the r.e. set we are looking at, k_i for $i \in \omega$ the number of corrections at the i-th r.e. set, and m_i for $i \in \omega$ the length of the initial segment of X currently meeting the i-th r.e. set (0 if not yet met). We start with $\psi = \emptyset$, $X = \langle\rangle$, $r = 0$, and $k_i, m_i = 0$ for all i.

At stage $n+1$ we search simultaneously for i such that $m_i = 0$, $\tau \supseteq X_n | \max_{j<i}(m_j)$, and s to find $\{i\}^{\tau}_{s}\downarrow$. If $i > r$ we have found a new r.e. set and add it by letting $m_i = \text{length}(\tau)$, $X_{n+1} = \tau$, $r = i$, and for l such that $\text{length}(X_n) < l \leq \text{length}(X_{n+1})$ we let $\psi(f(l) - k_l, l) = X_{n+1}(l)$. If instead $i < r$ we have found a r.e. set we have skipped over and restart at that point. We do this by first setting m_i =length(τ) and for l such that $\max_{j<i}(m_j) < l \leq m_r$ setting k_l to $k_l + 1$. We next reset X_{n+1} to τ (it will not extend X_n, but will extend $X_n | \max_{j<i}(m_j))$. Finally, for l such that $\max_{j<i}(m_j) < l \leq m_i$ we extend ψ by letting $\psi(f(l) - k_l, l) = X_{n+1}(l)$ and then for j with $i < j \leq r$ we set $m_j = 0$.

We note that the values of X used to meet the nth r.e. set are changed at most n times, once for every earlier r.e. set that was skipped over and discovered later. The function f bounds the number of corrections needed, and $\psi(b, n)$ witnesses X is ω-r.e. by starting with $b = f(n)$ and moving b down one every time a correction is made. □

We note that by the REA Completeness Theorem (Jockusch and Shore [8]) this gives that for every $X \geq_T \emptyset^{(\omega)}$ there are sets A and J such that $X \equiv_T A \oplus J$ where J is 1-generic (A).

Chapter 4

Relatively hyperimmune-free reals

4.1 Co-Countably Many Reals

We show that all but countably many reals are relatively hyperimmune-free by using the theorem of Reimann and Slaman [17] as in Section 3.1.

Theorem 4.1.1. *The set of reals which are not relatively hyperimmune-free is countable*

Proof. Let $\mathcal{B} = \{(X, Z) \mid \exists M <_T X\, [Z \leq_T M \ \wedge \ \forall f \leq_T M \oplus X\, \exists g \leq_T M\, [g \text{ dominates } f]\}$. We note $\mathcal{B}$ is arithmetic ($M <_T X$) hence Borel. Let C and Z be arbitrary reals. Let $M = C \oplus Z$ and let Y be a Spector minimal cover of M. Then $C \leq_T Y$ and M witnesses that $Y \in \mathcal{B}^Z$. Hence for every Z we have $\mathcal{B}^Z$ cofinal in the Turing degrees. Thus by the theorem of Reimann and Slaman [17] for all but countably many reals X there exists reals Y and G such that $X \equiv_{T,G} Y$ and $Y \in \mathcal{B}^G$.

Let M witness that $Y \in \mathcal{B}^G$. We wish to show that M witnesses that X is relatively hyperimmune-free. Suppose $X \leq_T M$. Then since $G \leq_T M$ we have $X \oplus G \leq_T M$

so $Y \leq_T M$ for a contradiction. Hence $X \not\leq_T M$. Let $f \leq_T X \oplus M$ be arbitrary. Since $G \leq_T M$ we have $X \oplus M \leq_T Y \oplus M$ so $f \leq_T Y \oplus M$. Since $Y \in \mathcal{B}^G$ there is a $g \leq_T M$ such that g dominates f. Thus M witnesses that X is relatively hyperimmune-free. Therefore, all but countably many reals are relatively hyperimmune-free. □

4.2 Iterated Hyperjumps

We begin by showing all reals of α-REA degree are not relatively hyperimmune-free. We see that an r.e. real W_n cannot be relatively hyperimmune-free since W_n can compute the function f where $f(m) = 0$ if $m \notin W_n$ and $f(m)$ is the least s such that $m \in W_{n,s}$ otherwise. Any function g which dominates f can then compute W_n by $m \in W_n \leftrightarrow m \in W_{n,g(m)}$. Our proof will use this procedure inductively.

Lemma 4.2.1. *Let $\alpha < \omega_1^{CK}$ and X be a real of α-REA degree. Then X is not relatively hyperimmune-free.*

Proof. Suppose not, witnessed by M. Let S be a system of notations for α and let h be a recursive function such that $(X)_n = W_{h(l)}^{(X)_l} \oplus (X)_l$ for all n, l such that $S(n) = S(l) + 1$. We define a function f by $f(\langle m, n\rangle)$ is the least z such that $m \in W_{h(k),z}^{(X)_k}$ if for some β we have $S(k) = \beta$, $S(n) = \beta + 1$, and $m \in (X)_n$. We let $f(\langle m, n\rangle) = 0$ otherwise. We note $f \leq_T X$ so there is some $g \leq_T M$ such that g dominates f.

We will show by induction that for all $\beta \leq \alpha$ and l such that $S(l) = \beta$ we have $(X)_l \leq_T g$ uniformly. For successor ordinals, let l, n be such that $S(n) = S(l) + 1$ and $(X)_l \leq_T g$ uniformly. Let Z be such that $(X)_n = Z \oplus (X)_l$. Then $m \in Z \leftrightarrow m \in W_{h(l),g(\langle m,n\rangle)}^{(X)_l}$. For limit ordinals, let λ, b be such that $S(b) = \lambda$ and for all l with $S(l) < \lambda$

we have $(X)_l \leq_T g$ uniformly. Then $\langle p, l\rangle \in (X)_b \leftrightarrow p \in (X)_l$ and $S(l) < b$.

By the induction, $X \leq_T g$ so $X \leq_T M$ for a contradiction. □

We note that the above proof can be relativized to any Y which is not relatively hyperimmune-free. Hence if X is relatively hyperimmune-free and $Y \leq_T X \leq_T Y'$ then Y is relatively hyperimmune-free.

We will use an inductive procedure to show iterated hyperjumps are not relatively hyperimmune-free, as in section 3.3. We will also continue to view $\mathcal{O}$ as $\{e \mid U_e$ is well founded $\}$ where U_e denotes the eth recursive tree in $\omega^{<\omega}$. We will show $\mathcal{O}$ is not relatively hyperimmune-free by having it compute a function which bounds how far to the right a path through U_e can be.

Lemma 4.2.2. *$\mathcal{O}$ is not relatively hyperimmune-free.*

Proof. Suppose not, witnessed by M. For $e \notin \mathcal{O}$, let τ_e denote the leftmost path through U_e. We note $\tau_e \leq_T \mathcal{O}$ since $\mathcal{O}$ can trace subtrees to construct τ_e. Define the function f by $f(\langle e, n\rangle) = \tau_e(n)$ if $e \notin \mathcal{O}$ and $f(\langle e, n\rangle) = 0$ otherwise. We note $f \leq_T \mathcal{O}$ so let $g \leq_T M$ dominate f.

Let $U_e|g = \{\sigma \in U_e \mid \forall n < \text{length}(\sigma)\, [\sigma(n) < g(\langle e, n\rangle)]\}$. Define the function h by $h(e)$ is the least z such that $U_e|g$ contains no strings of length z if $e \in \mathcal{O}$ and $h(e) = 0$ otherwise. We note $h \leq_T \mathcal{O} \oplus M$ so let $j \leq_T M$ dominate h. Then $e \in \mathcal{O}$ if and only if $U_e|g$ contains a string of length $j(e)$. Hence $\mathcal{O} \leq_T g \oplus j$ so $\mathcal{O} \leq_T M$ for a contradiction. □

Corollary 4.2.3. *Let X be a real such that $\mathcal{O}^X$ is relatively hyperimmune-free, witnessed by M. Then $X \not\leq_T M$.*

Proof. If $X \leq_T M$ then we could apply the above lemma relative to X and obtain a contradiction. □

Corollary 4.2.4. *Let α be a real such that $\mathcal{O}^{(\alpha)}$ is not relatively hyperimmune-free. Then $\mathcal{O}^{(\alpha+1)}$ is not relatively hyperimmune-free.*

Proof. Suppose not, witnessed by M. By the previous Corollary, $\mathcal{O}^{(\alpha)} \not\leq_T M$. But then M witnesses that $\mathcal{O}^{(\alpha)}$ is relatively hyperimmune-free, for a contradiction. □

Corollary 4.2.5. *The statement "All but countably many reals are relatively hyperimmune-free" fails to hold in Π^1_1-CA.*

Proof. We can use the previous Corollary to show by induction that $\mathcal{O}^{(n)}$ is not relatively hyperimmune-free for all $n \in \omega$. We can then apply the proof of Corollary 3.3.4. □

We will now show limit ordinal iterated hyperjumps are not relatively hyperimmune-free. The proof is analogous to that of Lemma 4.2.2.

Lemma 4.2.6. *Let o be a system of notations for a limit ordinal α such that $o \leq_T \mathcal{O}^{(\gamma)}$ for some $\gamma < \alpha$ and for all $\beta < \alpha$ we have $\mathcal{O}^{(\beta)}$ is not relatively hyperimmune-free. Then $\mathcal{O}^{(\alpha)}$ is not relatively hyperimmune-free.*

Proof. Suppose not, witnessed by M. We note that $\mathcal{O}^{(\beta)} \leq_T M$ for all $\beta < \alpha$ since otherwise M would witness $\mathcal{O}^{(\beta)}$ being relatively hyperimmune-free. Since $\gamma < \alpha$, we have $o \leq_T M$.

Let τ_e^X denote the leftmost path through U_e^X for $e \notin \mathcal{O}^X$. We define a function f. Let n, l be such that $o(n) = o(l) + 1$ (if $o(n) = 0$ we use $\emptyset$ for $\mathcal{O}^{(o(l))}$). Let $f(\langle n, e, m\rangle) = \tau_e^{\mathcal{O}^{(o(l))}}(m)$ if $e \notin \mathcal{O}^{(o(n))}$ and $f(\langle n, e, m\rangle) = 0$ otherwise. For n such that $o(n)$ is a limit ordinal, let $f(\langle n, e, m\rangle) = 0$. Since $f \leq_T \mathcal{O}^{(\alpha)}$ let $g \leq_T M$ dominate f.

Let $U_e^n|g = \{\sigma \in U_e^{\mathcal{O}^{(o(n))}} \mid \forall l < \text{length}(\sigma)\, [\sigma(l) < g(\langle n, e, l\rangle)]\}$. We next define the function h. If $o(n)$ is a limit ordinal then $h(\langle n, e\rangle) = 0$. If n, l are such that $o(n) = o(l) + 1$ then $h(\langle n, e\rangle)$ is the least z such that $U_e^l|g$ contains no strings of length z if $e \in \mathcal{O}^{(o(n))}$ and $h(\langle n, e\rangle) = 0$ otherwise. Again, $h \leq_T M \oplus \mathcal{O}^{(\alpha)}$ so let $j \leq_T M$ dominate h.

We wish to show $\mathcal{O}^{(\alpha)} \leq_T g \oplus j \oplus o$ so that $\mathcal{O}^{(\alpha)} \leq_T M$ for a contradiction. Let $\beta \leq \alpha$ be arbitrary. If $o(n)$ is a limit ordinal then $\langle n, m, e\rangle \in \mathcal{O}^{(\beta)}$ if and only if $e \in \mathcal{O}^{(o(m))}$. If $o(n) = o(l) + 1$ then $\langle n, e\rangle \in \mathcal{O}^{(\beta)}$ if and only if $U_e^n|g$ contains a string of length $j(\langle n, e\rangle)$. In either case, finitely many queries to $\mathcal{O}^{(\delta)}$ for some $\delta < \beta$ are used. These queries can then be answered by further use of this procedure. By the well-foundedness of the ordinals, we can uniformly determine if any $m \in \mathcal{O}^{(\alpha)}$ by finitely many queries to j, g, o. Hence $\mathcal{O}^{(\alpha)} \leq_T g \oplus j \oplus o$, as desired. □

We can now apply induction as in section 3.3.

Theorem 4.2.7. *Let λ be the least ordinal such that $\sup_{\beta<\lambda}(\beta$th admissible$) = \lambda$. Then for all $\alpha < \lambda$ we have $\mathcal{O}^{(\alpha)}$ is not relatively hyperimmune-free.*

Proof. Suppose not. Let $\beta < \lambda$ be the least ordinal such that $\mathcal{O}^{(\beta)}$ is relatively hyperimmune-free. Let f be the function used in Theorem 3.3.6. Then $\beta < f(\beta)$. By Corollary 4.2.4, β is a nonzero limit ordinal so choose $\gamma < \beta$ such that $\beta < f(\gamma)$. Then $\beta < \omega_1^{\mathcal{O}^{(\gamma)}}$ so we can fix a system of notations, o, for β recursive in $\mathcal{O}^{(\gamma+1)}$. Then by Lemma 4.2.6 we have $\mathcal{O}^{(\beta)}$ is not relatively hyperimmune-free, for a contradiction. □

Chapter 5

Conclusion

While they differ at low levels of complexity, the sets of reals which are random relative to some continuous measure, generic relative to some perfect tree, and relatively hyperimmune-free are remarkably similar. This raises the questions of what other sets of relative reals follow this pattern and if there is a deeper explanation. Woodin has shown that the co-countability of reals random relative to some continuous measure and reals generic relative to some perfect tree can be proved equivalent above a base theory which proves neither outright [22].

How much of ZFC is needed to prove all but countably many reals are relatively hyperimmune-free is not yet known. There is also still considerable room left to explore in determining which reals are 1-generic relative to some perfect tree. In particular, it is not yet known if every real not 1-generic relative to any perfect tree is hyperarithmetic. Finally, there are many properties related to relative recursive enumerability that may be worth studying. Among these are the sets of reals which are relatively d.r.e. low, semi-low, semicreative, non-cappable, atomless, and maximal.

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