Relatively r.e. reals

Bernard A. Anderson

University of California, Berkeley

October 28, 2006

www.math.berkeley.edu/~ baander

Introduction

Relative Properties

One area of study in Recursion Theory is which reals X hold some property relative to another real Y.

Examples: For which reals X does there exist a Y such that

- X is properly r.e. (or Σ_n) in Y
- X is n-generic in Y
- X is n-random in Y

For these questions, we work at the level of reals, not of Turing degrees.

Motivation

Relative Properties (continued)

These investigations are interesting when reals with traits quite different from some property, still have this property in some context.

For example, a real X of minimal degree can be (properly) r.e. relative to some Y or "n-generic" relative to some Y.

I have previously worked on relative genericity and classifying which reals are n-generic relative to a perfect tree

Motivation (continued)

Relative Recursive Enumerability

On the other side of the scale is the question:

When is a real r.e. relative to another real?

We find the the set of relatively r.e. reals is "as large as possible."

In addition, the witness that a real X is relatively r.e. can be found uniformly from X'.

Definitions

Definition

A real X is relatively r.e. if there exists a real Y such that X is r.e. (Y) and $X \not\leq_T Y$.

Definition

A real X is relatively REA if there exists a real $Y <_T X$ such that X is r.e. (Y).

Prior Results

Theorem (Jockusch)

Every 1-generic real is relatively REA.

Theorem (Kurtz)

The set of relatively REA reals has measure one.

Theorem (Kautz)

Every 2-random real is relatively REA.

Prior Results (continued)

Reals that are not relatively r.e.

We note that every Turing degree contains a real which is not relatively r.e.

Given a real X, we define Y by

$Y = \{\langle\sigma\rangle \mid \sigma \subseteq X\}$

where $\langle\rangle$ is some canonical map from $2^{<\omega} \to \omega$.

Then $Y \equiv_T X$ and for any Z such that Y is r.e. (Z) we have that $\overline{Y}$ is r.e. (Z) by

$n \in \overline{Y} \Leftrightarrow \exists\langle\sigma\rangle \in Y[n \notin \sigma]$.

Σ_1 Reductions on Enumerations

Reals that are not relatively r.e. (continued)

In general, a real is not relatively r.e. any time there is a Σ_1 process which takes every enumeration of the real to an enumeration of the complement.

Σ_1 Reductions on Enumerations (continued)

Definition
$A \leq_e B$ if there is a Σ_1 machine which given an enumeration of B in any order, outputs an enumeration of A.

Definition
$A \leq_{e_n} B$ if there exists a Σ_n set C such that $m \in A$ iff $\exists \langle m, E \rangle \in C \, [E \subseteq B]$

We note that $A \leq_{e_1} B \Leftrightarrow A \leq_e B$.

Main Result

Theorem

$\overline{X} \not\leq_{e_1} X \Leftrightarrow X$ *is relatively r.e.*

Nature of R.E. Sets

Strategy for Proof

Let X be a real such that $\overline{X} \not\leq_{e_1} X$.

We wish to find a real $Y \not\geq_T X$ such that X is r.e. (Y).

By definition a set is r.e. if its elements can be enumerated.

We will choose a set Y which is a list of the elements of X.

X is then clearly r.e. (Y).

We need to find an order for the list such that $Y \not\geq_T X$.

Enumeration Order

Partial Order

We define a partial order $\mathbb{P}_X$ of enumerations of elements of X.

$$\mathbb{P}_X = \left(\{\sigma \in \omega^{<\omega} \mid \forall n < \text{length}(\sigma)[\sigma(n) \in X]\}\, ,\, \supseteq \right)$$

We will use a generic for $\mathbb{P}_X$ to witness that X is relatively r.e.

Monadic notation

We define $m : \omega^{<\omega} \to 2^{<\omega}$ to be the function for monadic notation.

$m(\sigma) = 1^{\sigma(0)}\hat{}0\hat{}1^{\sigma(1)}\hat{}0\hat{}1^{\sigma(2)} \ldots 0\hat{}1^{\sigma(n)}$ where $n =$length(σ).

Proof of Main Result

Theorem

$\overline{X} \not\leq_{e_1} X \Leftrightarrow X$ *is relatively r.e.*

Proof

($\Longleftarrow$) Let Y witness X is relatively r.e. and suppose $\overline{X} \leq_e X$.

Given Y we can enumerate X and use the fact that $\overline{X} \leq_e X$ to enumerate $\overline{X}$.

Hence $\overline{X}$ is r.e. (Y) so $X \leq_T Y$ for a contradiction. Therefore $\overline{X} \not\leq_e X$.

Proof of Main Result (continued)

Proof (continued)

($\Longrightarrow$) $\overline{X} \not\leq_{e_1} X$. Let G be a 1-generic (X) real through $\mathbb{P}_X$ and let $Y = m(G)$.

X is r.e. (Y) since $n \in X \Leftrightarrow 0\hat{\ }1^n\hat{\ }0 \subseteq Y$.

Suppose towards a contradiction that $\overline{X}$ is r.e. (Y). Let $\overline{X} = W_k^Y$.

Proof of Main Result (continued)

Proof (continued)

We now use genericity to show that any enumeration of X extending some condition computes an enumeration of $\overline{X}$.

Let $S = \{\sigma \in \mathbb{P}_X \mid \exists n \in X\,[n \in W_k^{m(\sigma)}]\}$.

$G \notin S$. Since G is 1-generic (X), there is a $q \in G$ such that $\forall r \leq_{\mathbb{P}_X} q\,[r \notin S]$.

Proof of Main Result (continued)

Proof (continued)

Let $Q = \{p \in \mathbb{P}_X \mid p \leq_{\mathbb{P}_X} q\}$.

Then for all $p \in Q$ and $n \in \omega$ we have $n \in W_k^{m(p)} \Rightarrow n \in \overline{X}$.

Conversely, if $n \in \overline{X}$ then $n \in W_k^Y$ so there is a $p \in Q$ such that $n \in W_k^{m(p)}$.

Hence $n \in \overline{X} \Leftrightarrow \exists p \in Q\,[n \in W_k^{m(p)}]$. $\quad (*)$

Proof of Main Result (conclusion)

Proof (continued)

Given an enumeration of X we can generate an enumeration of Q by adding elements of X to q in all possible orders.

We can then find an enumeration of $\overline{X}$ from the enumeration of Q using $(*)$.

Hence $\overline{X} \leq_e X$ for a contradiction.

Therefore $\overline{X}$ is not r.e. (Y) and Y witnesses X is relatively r.e. $\square$

Observations

Arithmetic Form

The proof can be done in an arithmetic context using an ordinary 1-generic (X) real G and letting

$Y = \{\langle n, m\rangle \mid n \in X \wedge \langle n, m\rangle \in G\}$.

Bounds on Witness

From the proof we see that if $\overline{X} \not\leq_{e_1} X$ then a witness Y that X is relatively r.e. can be found uniformly from any real which is 1-generic (X)

For example, for any real $Z >_T X$ such that Z is r.e. (X) there is a $Y \leq_T Z$ such that Y witnesses X is relatively r.e.

Corollaries

Corollary

$\overline{X} \not\leq_{e_n} X \Leftrightarrow X$ *is relatively* Σ_n.

Proof

We observe $\overline{X} \not\leq_{e_n} X$ implies $\overline{X} \not\leq_{e_n} X \oplus 0^{(n)}$. By the proof of the main Theorem $\exists Z\,[X \oplus 0^{(n)}$ is r.e. (Z) and $\overline{X}$ is not r.e. $(Z)]$.

We note $0^{(n-1)} \leq_T Z$ so by the Friedberg Inversion Theorem let Y be such that $Y^{(n-1)} \equiv_T Z$.

Then X is r.e. $(Y^{(n-1)})$ so X is $\Sigma_n(Y)$.

$\overline{X}$ is $\Sigma_n(Y)$ implies $\overline{X}$ is r.e. (Z) for a contradiction. $\square$

Corollaries (continued)

Corollary

$\overline{X} \not\leq_{e_1} X \Rightarrow \exists Y\,[X \textit{ is r.e. } (Y) \textit{ and } X \oplus Y \equiv_T Y']$.

Proof

Since $\overline{X} \not\leq_{e_1} X$, let $Z \not\geq_T X$ be such that X is r.e. (Z).

We use the proof of the Posner-Robinson theorem above Z to find a Y such that $X \oplus Y \equiv_T Y'$ and $Y \geq_T Z$.

Then X is r.e. (Y). $\square$

Relatively Simple And Above Reals

Definition
A real X is relatively simple and above if there is a real $Y <_T X$ such that X is r.e. (Y) and for every infinite $Z \subseteq \overline{X}$ we have that Z is not r.e. (Y).

We note relatively simple and above implies relatively REA.

Relatively Simple And Above Reals (continued)

Alternate proof that every 1-generic is relatively REA

We can use techniques from the arithmetic form of the proof of the main theorem to prove a slightly stronger version of Jockusch's theorem that every 1-generic is relatively REA.

For our witness we will use an enumeration of X in an order given by X itself.

1-generic reals are relatively simple and above

Theorem

Let X be 1-generic. Then X is relatively simple and above.

Proof

Let $Y = \{\langle n, m\rangle \mid n \in X \wedge \langle n, m\rangle \notin X\}$.

Then X is r.e. (Y) and $Y \leq_T X$.

Let Z be an arbitrary infinite subset of $\overline{X}$.

It remains to show that Z is not r.e. (Y).

Suppose towards a contradiction that Z is r.e. (Y). Let $Z = W_k^Y$.

1-generic reals are relatively simple and above (continued)

Proof (continued)

Define a function $o : 2^{<\omega} \to 2^{<\omega}$ by
$o(\sigma(\langle n, m\rangle)) = 1$ iff $\sigma(n) = 1$ and $\sigma(\langle n, m\rangle) = 0$.
So $Y = o(X)$ and $Z = W_k^{o(X)}$.

Since $X \cap W_k^{o(X)} = \emptyset$, and X is 1-generic, we find a condition where for every extension τ we have $\tau \cap W_k^{o(\tau)} = \emptyset$.

We then get a contradiction by adding an element to τ without changing $o(\tau)$.

1-generic reals are relatively simple and above (continued)

Proof (continued)

Let $S = \{\sigma \mid \exists n\, [n \in \sigma \ \wedge \ n \in W_k^{o(\sigma)}]\}$.

$X \notin S$. Since X is 1-generic, for some l for every τ extending $X|l$ we have

$\tau \notin S$ so $\forall n\, [n \in W_k^{o(\tau)} \rightarrow n \notin \tau]$.

Let $p \in Z$ with $p > l$. Let $t > l$ be such that $p \in W_k^{o(X|t)}$.

For any $\sigma \supseteq X|l$ such that $o(\sigma) = o(X|t)$ we have $p \in W_k^{o(\sigma)}$ so $p \notin \sigma$.

1-generic reals are relatively simple and above (continued)

Proof (continued)

We obtain a contradiction by finding a $\sigma \supseteq X|l$ such that $o(\sigma) = o(X|t)$ and $p \in \sigma$.

Let $\sigma_0 = X|t$ and $\sigma_1 = \sigma_0$ except $\sigma_1(p) = 1$.

At stage $i + 1$, we consider all b such that $\sigma_i(b) \neq \sigma_{i-1}(b)$.
For every a such that $\sigma_i(\langle b, a\rangle) = 0$ we let $\sigma_{i+1}(\langle b, a\rangle) = 1$.
For all other values, we let $\sigma_{i+1} = \sigma_i$.

1-generic reals are relatively simple and above (conclusion)

Proof (continued)

Since $\langle b, a \rangle > a$ and $X|t$ is finite, this procedure halts. We let σ be the final value.

Then σ is as desired for a contradiction. Hence Z is not r.e. (Y) and X is relatively simple and above. $\square$

Other Results

Lemma

$\overline{X} \not\leq_{e_1} X \Rightarrow$
$\exists Y\,[X$ is r.e. (Y) and $X \not\leq_T Y$ and Y is not arithmetic in $X]$.