A bounded jump for the bounded Turing degrees

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May 27, 2010

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Introduction

Bounded Turing degrees

In this talk we will work with the bounded Turing degrees.

Definition

$A \leq_{bT} B$ if there is a Turing reduction Φ and a computable function f such that $A(n) = \Phi^{B \upharpoonright f(n)}(n)$ for all n.

Weak truth-table degrees

The bounded Turing degrees are often called the weak truth-table degrees (denoted $\leq_{wtt}$).

Introduction (continued)

The Turing jump on the bounded Turing degrees

The Turing jump often works differently on the bounded Turing degrees than on the Turing degrees.

For example, Shoenfield proved for every Σ_2 set B there is a $A \leq_T \emptyset'$ such that $A' \equiv_T B$.

However Csima, Downey, and Ng proved there is a Σ_2 set $C >_{tt} \emptyset'$ such that for all $D \leq_T \emptyset'$ we have $D' \not\equiv_{bT} C$.

Motivation

Finding a bounded jump

Can we find a "bounded" jump operator which corresponds to the definition of the bounded Turing degrees?

We would want such an operator to interact with the bounded Turing degrees in a manner analogous to the interaction of the Turing jump with the Turing degrees.

Motivation (continued)

Desired properties

- Limited use of oracle
- Equivalent to similar operators
- Strictly increasing
- Order preserving
- Distinct from known operators

Definition

Bounded jump

We define the bounded jump.

Definition

$A^b = \{x \mid \exists\, i < x[\varphi_i(x)\downarrow \ \wedge \Phi_x^{A\upharpoonright \varphi_i(x)}(x)\downarrow]\}.$

We let A^{nb} denote the n-th bounded jump.

Similar operators

A more general form

The bounded jump is equivalent to a more general form.

Definition

$A^{b_0} = \{\langle e, i, j \rangle \mid \varphi_i(j)\downarrow \ \wedge \Phi_e^{A \upharpoonright \varphi_i(j)}(j)\downarrow\}.$

Theorem

1. $A^{b_0} \leq_1 A^b$
2. $A^b \leq_{tt} A^{b_0}$
3. *There exists A such that $A^b \not\leq_1 A^{b_0}$*

Similar operators (continued)

A simple form

A simplified form does not work as a jump operator.

Definition

$A^i = \{x \mid \Phi_x^{A \upharpoonright x}(x)\downarrow\}$

Remark

Let $A \geq_{bT} \emptyset'$. *Then* $A^i \leq_{bT} A$.

Properties

Basic properties

1. $\emptyset^b \equiv_1 \emptyset'$
2. $A \leq_1 A^b$
3. $A^b \leq_1 A'$ (since A^b is c.e.(A))

Properties (continued)

Strictly increasing

Theorem

$A^b \not\leq_{bT} A$

Order preserving

Theorem

$A \leq_{bT} B \Rightarrow A^{b_0} \leq_1 B^{b_0}$

Corollary

1. $A \leq_{bT} B \Rightarrow A^b \leq_{tt} B^b$
2. $\emptyset' \leq_{tt} A^b$

Properties (continued)

A^b and A'

Proposition

$A^b \equiv_T A \oplus \emptyset'$

Corollary

1. *If* $A' \not\leq_T A \oplus \emptyset'$ *then* $A' \not\leq_T A^b$
2. *If* $A \geq_T \emptyset'$ *then* $A^b \equiv_T A$

Properties (continued)

A^b and $A \oplus \emptyset'$

Since the bounded jump is strictly increasing, if $A \geq_T \emptyset'$ then $A^b \not\equiv_{bT} A \oplus \emptyset'$

Theorem

Let A be 3-random. Then $A^b \not\leq_{bT} A \oplus \emptyset'$.

It follows that the class of A such that $A^b \equiv_{bT} A \oplus \emptyset'$ has measure zero.

Properties (continued)

Strong jump inversion

Anderson proved that for every $A \geq_{tt} \emptyset'$ there is a B such that $B \oplus \emptyset' \equiv_{tt} B' \equiv_{tt} A$.

As a result, for every $A \geq_{tt} \emptyset^b$ there is a B such that $B \oplus \emptyset^b \equiv_{tt} B^b \equiv_{tt} A$.

A similar analysis gives that for every $A \geq_{bT} \emptyset'$ there is a B such that $B \oplus \emptyset^b \equiv_{bT} B^b \equiv_{bT} A$.

Main results

Ershov Hierarchy

The Turing jump characterizes the arithmetic hierarchy. We will see that the bounded jump characterizes the Ershov hierarchy.

Definition

A is α-c.e. for $\alpha \geq \omega$ if there is a partial computable $\psi : \omega \times \alpha \to \{0, 1\}$ such that for all n there is a γ such that $\psi(n, \gamma)\downarrow$ and for the least such γ we have $A(n) = \psi(n, \gamma)$.

Definition

A is a *tt*-cylinder if for all B we have $B \leq_{tt} A \Rightarrow B \leq_1 A$.

Main results (continued)

Ershov Hierarchy (continued)

Theorem (Folklore)

$A \leq_{bT} \emptyset' \Leftrightarrow A$ *is* ω *c.e.* $\Leftrightarrow A \leq_{tt} \emptyset'$

Theorem

For $n \geq 2$*, we have* $A \leq_{bT} \emptyset^{nb} \Leftrightarrow A$ *is* ω^n*-c.e.* $\Leftrightarrow A \leq_1 \emptyset^{nb}$

Corollary

For $n \geq 2$*, we have that* $\emptyset^{nb}$ *is a tt-cylinder.*

Main results (continued)

Ideas in proof

Lemma

1. *If $B \leq_{bT} A$ and A is ω^n-c.e. then B is ω^n-c.e.*
2. *If A is ω^n-c.e. then A^b is ω^{n+1}-c.e.*

For each of these we let ψ witness A is ω^n-c.e. and build a larger χ which level by level estimates the result based on the values in ψ.

In 2. if our bound estimate changes, we drop a ω^n level and start over.

Main results (continued)

Ideas in proof (continued)

Lemma

If A is ω^2-c.e. then $A \leq_1 \emptyset^{2b}$

We cannot bound in advance the amount of $\emptyset^b$ required to answer all the Σ_1 questions needed to determine if $n \in A$.

However, we can bound in advance the indices of computable functions which bound the amount of $\emptyset^b$ used.

A is ω^n-c.e. $\Rightarrow A \leq_1 \emptyset^{nb}$ for $n > 2$ is proved by induction.

Main results (continued)

Shoenfield inversion

We noted earlier that Shoenfield inversion fails to hold for the bounded Turing degrees with the Turing jump.

However, Shoenfield inversion does hold for the bounded Turing degrees with the bounded jump.

Theorem

Given B such that $\emptyset^b \leq_{bT} B \leq_{bT} \emptyset^{2b}$ there is a $A \leq_{bT} \emptyset^b$ such that $A^b \equiv_{bT} B$

Main results (continued)

Ideas in proof

Let ψ witness B is ω^2-c.e. We build A to be ω-c.e.

Pick g by the Recursion Theorem so that its values are far apart. g will witness $B \leq_1 A^b$.

Choose markers x_n^i to represent the least i such that $\psi(n, \omega \cdot i + j)\downarrow$ for some j.

Main results (continued)

Ideas in proof (continued)

Declare $\Phi_{g(n)}^{A \cup \{x_n^i\} \upharpoonright x_n^i}(g(n))\downarrow$ and vary $A_s(x_n^i)$ to match $B_s(n)$.

If $\psi(n, \omega \cdot \tilde{\imath} + j)\downarrow$ for a lower $\tilde{\imath}$, then move to new marker $x_n^{\tilde{\imath}}$.

Construction is computable, and marker locations bounded. Use this to show $A^b \leq_{bT} B$.

Conclusion

Further progress

We can determine if other theorems about the Turing jump hold for the bounded jump.

For example, Sacks showed that for Σ_2 sets $B \geq_T \emptyset'$ there is a c.e. set A such that $A' \equiv_T B$.

Csima, Downey, and Ng proved Sacks jump inversion fails for the bounded Turing degrees with the Turing jump.

Not yet known if Sacks jump inversion holds for the bounded Turing degrees with the bounded jump.

Conclusion (continued)

Other open areas

Definition

A is bounded high if $A^b \geq_{bT} \emptyset^{2b}$. A is bounded low if $A^b \leq_{bT} \emptyset^b$.

We can attempt to characterize which sets are bounded high or bounded low. We can also look at other definitions using the bounded jump.

Gerla developed jump operators for the truth-table and bounded truth-table degrees. Not much work has been done with these operators yet.