# Automorphisms of the truth-table degrees

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# Introduction

## Degree structures and automorphisms

A basic question of Computability Theory is the structure of the Turing Degrees and other computational degrees.

One way we can study these structures is to examine their automorphisms.

The automorphisms of the Turing degrees, many-one degrees, and hyperdegrees have been studied extensively.

# Introduction (continued)

## Truth-table degree automorphisms

In comparison, the automorphisms of the truth-table degrees remain relatively unexplored.

We show that every automorphism of the truth-table degrees is fixed on some cone.

We also consider the possibility of further progress in studying automorphisms of the truth-table degrees.

# Definitions

### Reals

We view reals as infinite binary strings.

### Definition

A reduction $\Phi$ is a truth-table reduction if there is a computable function $f$ such that for all $n$, for every string $\sigma$ of length $f(n)$, we have $\Phi^{\sigma}(n)\downarrow$.

$A \leq_{tt} B$ if there is a truth-table reduction $\Phi$ such that $\Phi(B) = A$.

### Sets of degrees

We let $D$, $D_{tt}$, $D_m$, and $D_h$ denote the sets of Turing degrees, truth-table degrees, many-one degrees, and hyperdegrees, respectively.

# Definitions (continued)

## Definition

A bijection $\pi : D \rightarrow D$ is an automorphism if for all Turing degrees $x, y$ we have:

$x \leq_T y \Leftrightarrow \pi(x) \leq_T \pi(y)$.

Automorphisms of other degree structures are defined similarly.

## Definition

A real $G$ is $n$-generic if for every $\Sigma_n$ set $S$ of finite strings either there is an $l$ such that $G|l \in S$ or there is an $l$ such that for every $\tau \supseteq G|l$ we have $\tau \notin S$.

# Results

We prove the following results.

**Theorem**

*Let $X$ be a 2-generic real. Then $X' \not\leq_{tt} X \oplus 0'$.*

**Theorem**

*Let $X \geq_{tt} 0'$ be a real. Then there is a real $Y$ such that $Y \oplus 0' \equiv_{tt} Y' \equiv_{tt} X$.*

**Theorem**

*Let $\pi : D_{tt} \to D_{tt}$ be an automorphism. Then there is a degree $b$ such that for all $x \geq_T b$ we have $\pi(x) = x$.*

# Known results

## Spectrum

To gain a better understanding of the automorphisms of the truth-table degrees, we consider the automorphisms of other degree structures.

From results known so far, it seems the degree structures form a spectrum.

The stronger the reduction, the fewer the restrictions on the automorphisms.

## Hyperdegrees are rigid

There is no nontrivial automorphism of the hyperdegrees.

# Known results (continued)

## Many-one degrees

0 is the only m-degree fixed by every automorphism (Odifreddi).

There are $2^{2^\omega}$ many automorphisms of the m-degrees (Shore).

There is an automorphism of $D_m$ which is not fixed on any cone (Shore).

Combined with our main result, this provides a tangible difference between the automorphisms of the tt-degrees and the automorphisms of the m-degrees.

# Known results (continued)

## Turing degrees

Every automorphism of the Turing degrees is fixed on some cone (Nerode and Shore).

Every automorphism of the Turing degrees is fixed on the cone with base $0''$ (Slaman and Woodin).

Let $g \in D$ contain a 5-generic real. Then $\{g\}$ is an automorphism base for $D$ (Slaman and Woodin).

For all automorphisms $\pi$ and $\rho$ of $D$, if $\pi(g) = \rho(g)$ then $\pi = \rho$.

The Turing jump is definable in $D$ (Shore and Slaman).

$\pi(x') = (\pi(x))'$.

# Known results (continued)

## Are the Turing degrees rigid?

The set of automorphisms of the Turing degrees is at most countable (Slaman and Woodin).

The statement ”There is a nontrivial automorphism of the Turing degrees” is absolute between well-founded models of ZFC (Slaman and Woodin).

The Turing degrees are not rigid (Cooper).

The proof of this fact has not yet been verified by leading experts.

# Known results (conclusion)

## Truth-table degrees

There are few known results about automorphisms of the truth-table degrees.

For the structure of the truth-table degrees with jump, $(D_{tt}, \leq_{tt}, ')$, every automorphism is fixed on the cone with base $0^{(4)}$ (Kjos-Hanssen).

# Main result

We prove that every automorphism of $D_{tt}$ is fixed on some cone.

**Theorem**

*Let $\pi : D_{tt} \to D_{tt}$ be an automorphism. Then there is a degree $b$ such that for all $x \geq_{tt} b$ we have $\pi(x) = x$.*

# Towards proving the main result

## Adapting Nerode and Shore

Nerode and Shore noted that to adapt to the truth-table degrees their proof that every automorphism of $D$ is fixed on some cone required some form of jump inversion.

Mohrherr found that it sufficed to show strong jump inversion for the truth-table degrees:

For all reals $X \geq_{tt} 0'$ there is a real $Y$ such that

$Y \oplus 0' \equiv_{tt} Y' \equiv_{tt} X$.

# Jump inversions

Strong jump inversion holds for the Turing degrees.

## Theorem (Friedberg)

*Let $X \geq_T 0'$ be a real. Then there is a real $Y$ such that $Y \oplus 0' \equiv_T Y' \equiv_T X$.*

An analogue of the proof can be used to prove ordinary jump inversion for the truth-table degrees.

## Theorem (Mohrherr)

*Let $X \geq_{tt} 0'$ be a real. Then there is a real $Y$ such that $Y' \equiv_{tt} X$.*

# Jump inversions (continued)

Friedberg obtained strong jump inversion for the Turing degrees by using the following lemma.

### Lemma (Friedberg)

*Let $X$ be a 1-generic real. Then $X' \leq_T X \oplus 0'$.*

To obtain strong jump inversion for the truth-table degrees, we might hope to prove a similar lemma.

Somewhat unexpectedly, nearly the opposite situation holds.

### Theorem

*Let $X$ be a 2-generic real. Then $X' \not\leq_{tt} X \oplus 0'$.*

# $X$ is 2-generic $\Rightarrow X' \not\leq_{tt} X \oplus 0'$

## Idea for Proof

We wish to prove every 2-generic real $X$ is such that $X' \not\leq_{tt} X \oplus 0'$.

A truth-table computation of $X'$ is bounded in its use of $X$

However, the following is dense:

The $n$ required so that $\{e\}^{X \upharpoonright n}(e) \downarrow$ exceeds any fixed bound computable in $e$.

We use this to get a proof by contradiction.

# $X$ is 2-generic $\Rightarrow X' \not\leq_{tt} X \oplus 0'$—Proof

## Proof

Suppose not. Let $\Phi$ be a truth-table reduction with bound $f$ such that $\Phi(X \oplus 0') = X'$.

Without loss of generality, let $f(n)$ be even for all $n$.

We define a set $S$ of strings $\sigma$ which witness a failure of $\Phi(A \oplus 0') = A'$ for any $A \supset \sigma$.

# $X$ is 2-generic $\Rightarrow X' \not\leq_{tt} X \oplus 0'$—Proof (continued)

## Proof (continued)

Let $S = \{\sigma \in 2^{<\omega} \mid \exists n[\Phi^{(\sigma \oplus 0') \upharpoonright f(n)}(n) = 0 \;\wedge\; \{n\}^{\sigma}(n)\downarrow]\}$.

$X$ does not meet $S$ because $\Phi(X \oplus 0') = X'$.

$S$ is $\Sigma^0_1(0')$ so $S$ is $\Sigma^0_2$.

Since $X$ is 2-generic, there is an $l$ such that for every $\tau \supseteq X \upharpoonright l$ we have $\tau \notin S$.

# $X$ is 2-generic $\Rightarrow X' \not\leq_{tt} X \oplus 0'$—Proof (continued)

## Proof (continued)

Define $\{j(y)\}^\sigma(n) = 0$ if $\sigma(\frac{1}{2}[f(y)] + 1) = 0$ and $\{j(y)\}^\sigma(n)\uparrow$ otherwise.

By the Recursion Theorem, let $M$ be an infinite computable set such that $\{j(m)\} = \{m\}$ for all $m \in M$.

# $X$ is 2-generic $\Rightarrow X' \not\leq_{tt} X \oplus 0'$—Proof (continued)

## Proof (continued)

For $m \in M$, if $\{m\}^{\sigma}(m)$ converges depends only on the value of $\sigma(\frac{1}{2}[f(m)] + 1)$.

But for $\sigma \supseteq X \upharpoonright l$ the value of the jump is predicted based on $\sigma \upharpoonright \frac{1}{2}[f(m)]$.

By applying genericity again, we find an element of $M$ which can create a contradiction.

# $X$ is 2-generic $\Rightarrow X' \not\leq_{tt} X \oplus 0'$—Proof (continued)

**Claim**

$\exists v \in M$ such that $\frac{1}{2}[f(v)] > l$ and $X(\frac{1}{2}[f(v)] + 1) = 1$.

**Proof (claim)**

Let
$V = \{\sigma \in 2^{<\omega} \mid \exists m \in M[\frac{1}{2}[f(m)] > l \wedge \sigma(\frac{1}{2}[f(m)] + 1) = 1]\}$.

For any $\tau$ we have $\tau\hat{}1^j \in V$ for any large enough $j$.

By genericity, $X \upharpoonright k \in V$ for some $k$. Then let $v$ witness that $X \upharpoonright k \in V$. $\square$ (claim)

# $X$ is 2-generic $\Rightarrow X' \not\leq_{tt} X \oplus 0'$—Proof (continued)

## Proof (from claim)

Let $v$ be given by the claim.

Since $v \in M$ and $X(\frac{1}{2}[f(v)] + 1) = 1$ we have $\{v\}^X(v) = \{j(v)\}^X(v) = \uparrow$.

Hence $v \notin X'$ so $\Phi^{(X \oplus 0') \upharpoonright f(v)}(v) = 0$.

Let $\tau = (X \upharpoonright \frac{1}{2}[f(v)])\hat{\ }0$.

# $X$ is 2-generic $\Rightarrow X' \not\leq_{tt} X \oplus 0'$—Proof (conclusion)

## Proof (conclusion)

Since $\frac{1}{2}[f(v)] > l$ we have $\tau \supseteq X \upharpoonright l$ so $\tau \notin S$.

We already have $\Phi^{(X \oplus 0') \upharpoonright f(v)}(v) = 0$ so we must conclude $\{v\}^{\tau}(v)\uparrow$.

But $v \in M$ and $\tau(\frac{1}{2}[f(v)] + 1) = 0$ so $\{v\}^{\tau}(v) = \{j(v)\}^{\tau}(v) = 0$ for a contradiction. $\square$.

## Remark

Kjos-Hanssen has observed that Mohrherr's construction for $0'$ gives a 1-generic real $X$ such that $X' \leq_{tt} X \oplus 0'$. Hence this result is sharp.

# Towards proving strong jump inversion

## Difficulties

The proof of the previous theorem highlights the difficulties in proving strong jump inversion.

We must make many choices deciding the values for $Y'$ before we can extend $Y$ to ensure the choices are correct.

# Towards proving strong jump inversion (continued)

## Alternate approach

Our original proof solved this by using a “bushy” construction similar to those developed by Kumabe and Lewis.

Kučera noted that using a PA approach would result in a simpler proof.

## PA reals

PA is the set of (binary valued) diagonally noncomputable reals.

$$\mathrm{PA} = \{f \in 2^\omega \mid \forall x[f(x) \neq \{x\}(x)]\}.$$

PA can be used as a "universal" $\Pi^0_1$ class, which can easily code information.

# PA reals—Definitions

**Definition**

Let $f \in 2^\omega$ and let $M \subseteq \omega$ be an infinite set. Let $\langle m_i \mid i \in \omega \rangle$ be an increasing enumeration of $M$. We let $\mathrm{Restr}(f, M)$ denote the function $g$ such that $g(i) = f(m_i)$.

If we view $M$ as the set of coding locations, then $\mathrm{Restr}(f, M)$ is the real coded into $f$.

**Definition**

Let $\sigma \in 2^{<\omega}$. We define $\mathrm{Restr}(\sigma, M)$ to be the string $\tau$ such that $\tau(i) = \sigma(m_i)$ for all $i$ such that $m_i <$length$(\sigma)$.

**Definition**

Let $B \subseteq 2^\omega$. We define
$\mathrm{Restr}(B, M) = \{g \in 2^\omega \mid \exists f \in B\, [g = \mathrm{Restr}(f, M)]\}$.

# PA reals—Properties

## Theorem (Kučera and Slaman)

*Let $B \subseteq PA$ be a $\Pi^0_1$ class. Then there is an infinite computable set $M$ such that if $B \neq \emptyset$ then $Restr(B, M) = 2^\omega$. Furthermore, we can (uniformly) computably find an index for $M$ from an index for $B$.*

## Application

This theorem allows us to shrink the $\Pi^0_1$ class we are working in

As long as the resulting class is nonempty, it is still “thick” enough to code with.

# Tree notation

## Trees

Let $T \subseteq 2^{<\omega}$ be a tree. We let $[T]$ denote the $\Pi^0_1$ class of paths through $T$.

We let $T[\sigma]$ denote the part of $T$ compatible with the root $\sigma$.

$T[\sigma] = \{\tau \in T \mid \tau \supseteq \sigma \ \vee \ \tau \subseteq \sigma\}$.

## Indices

Let $W_n$ denote the $n$-th c.e. set of strings.

An index of a $\Pi^0_1$ tree $T$ is an $n$ such that $T = 2^{<\omega} \setminus W_n$.

An index for a $\Pi^0_1$ class $[T]$ is an index for $T$.

# Strong jump inversion—Proof

### Theorem

*Let $X \geq_{tt} 0'$ be a real. Then there is a real $Y$ such that $Y \oplus 0' \equiv_{tt} Y' \equiv_{tt} X$.*

### Proof

We build $Y$ in stages, using $\Pi^0_1$ trees $T_i$ for bookkeeping.

At each stage we have $Y_{i+1} \supseteq Y_i$, $T_{i+1} \subseteq T_i$, and $Y_i \in T_i$.

We start with $Y_0 = \langle\rangle$ and $T_0 =$PA.

# Strong jump inversion—Proof (continued)

## Proof (continued)

At stage $i$ we ask if there is a $d$ such that for all strings $\sigma \supseteq Y_i$ of length $d$ either $\sigma \notin T_i$ or $\{i\}^\sigma(i)\downarrow$.

This is a $\Sigma^0_1$ question which we use $0'$ to answer.

Essentially, we are asking if removing from $T_i$ the strings which force $i \in Y'$ results in an empty tree.

# Strong jump inversion—Proof (continued)

## Proof (continued)

If the answer is yes, we let $T_{i+1} = T_i[Y_i]$ and note $i \in Y'$ is forced.

If the answer is no, we let $T_{i+1} = T_i[Y_i] \setminus \{\sigma \in 2^{<\omega} \mid \{i\}^\sigma(i)\downarrow\}$ and note $i \notin Y'$ is forced.

In either case, $T_{i+1}$ is nonempty (by compactness and our choice of $Y_i$).

# Strong jump inversion—Proof (continued)

## Proof (continued)

Let $M$ be the set given by the theorem of Kučera and Slaman for $T$.

Let $m$ be the least element of $M$.

Using $0'$ we define $Y_i$ to be the leftmost string $\sigma \in T_{i+1}$ of length $m$ such that $\sigma(m) = X(i)$ and $\left[T_{i+1}[\sigma]\right] \neq \emptyset$.

$Y_i$ codes $X(i)$ at spot $m$. This completes stage $i$ of the construction.

# Strong jump inversion—Proof (continued)

## Proof (continued)

$X$ can follow the construction (since $X \geq_{tt} 0'$).

We can determine if $i \in Y'$ from the $i$-th stage of the construction. Hence $Y' \leq_T X$.

Similarly, given $Y$ and $0'$ we can follow the construction and read $X(i)$ from $Y(m)$.

Hence $Y \oplus 0' \equiv_T Y' \equiv_T X$. It remains to bound these computations.

# Strong jump inversion—Proof (continued)

## Proof (continued)

We define some computable functions. Let $m$ be an index of a $\Pi^0_1$ tree $T$.

$u(m,i)$ is an index for $T \cap \{\tau \in 2^{<\omega} \mid \{i\}^{\tau}(i)\uparrow\}$.

$q(m,\sigma)$ is an index for $T[\sigma]$.

$s(i)$ is the least element of the $M$ corresponding to the tree with index $i$ (for the theorem of Kučera and Slaman).

# Strong jump inversion—Proof (continued)

## Proof (continued)

We define computable functions $t$ and $l$ to bound the indices of $T_i$ and the lengths of $Y_i$

We start with $t(0) =$ index of PA and $l(0) = 0$.

To find $t(i+1)$ and $l(i+1)$ we take the largest possible result for any values used in the role of $Y' \upharpoonright i$, $X \upharpoonright i$, and $T_j$, $Y_j$ for $j \leq i$.

$$t(i+1) = \max_{\sigma \in 2^{\leq l(i)},\ e \leq t(i)} \Big( \max(q(e,\sigma), q(u(e,i),\sigma)) \Big)$$

$$l(i+1) = \max_{e \leq t(i+1)} (s(e))$$

# Strong jump inversion—Proof (continued)

## Proof (continued)

We next find a computable function $g$ such that stage $i$ of the construction requires at most $0' \upharpoonright g(i)$.

Let $h : 2^{<\omega} \times \omega \times \omega \to \omega$ be a computable function such that $h(\tau, n, i) \in 0'$ iff there is a number $d$ such that for all strings $\sigma$ of length $d$ extending $\tau$ we have $\sigma \in W_n$ or $\{i\}^{\sigma}(i)\downarrow$.

Let $j : 2^{<\omega} \times \omega \to \omega$ be a computable function such that $j(\tau, n) \in 0'$ iff there is a number $d$ such that for all strings $\sigma$ of length $d$ extending $\tau$ we have $\sigma \in W_n$.

$$g(i) = \max_{\tau \in 2^{\leq l(i+1)},\, m \leq i,\, n \leq t(i+1)} \Big( \max(h(\tau, n, m), j(\tau, n)) \Big)$$

# Strong jump inversion—Proof (conclusion)

## Proof (conclusion)

We now complete the proof.

To reach the $n$-th step of the construction requires $X \upharpoonright n$ and $0' \upharpoonright g(n)$.

Let $f$ witness $0' \leq_{tt} X$. Then to calculate $Y'(n)$ requires $X \upharpoonright \max(n, f(g(n)))$.

Similarly, to calculate $X(n)$ requires $Y \upharpoonright l(n+1)$ and $0' \upharpoonright g(n)$.

Therefore $Y \oplus 0' \equiv_{tt} Y' \equiv_{tt} X$. $\square$

# Multiple strong jump inversion

## Corollary

*Let $n \in \omega$ and let $X \geq_{tt} 0^{(n)}$ be a real. Then there is a real $Y$ such that $Y \oplus 0^{(n)} \equiv_{tt} Y^{(n)} \equiv_{tt} X$.*

## Proof

By induction. The base case is given by the theorem.

For the inductive case, assume the statement holds for $n$ and let $X \geq_{tt} 0^{(n+1)}$.

By the theorem relative to $0^{(n)}$ let $Z \geq_{tt} 0^{(n)}$ be such that $Z \oplus 0^{(n+1)} \equiv_{tt} Z' \equiv_{tt} X$.

# Multiple strong jump inversion (continued)

## Proof (continued)

By the inductive hypothesis for $Z$, let $Y$ be such that $Y \oplus 0^{(n)} \equiv_{tt} Y^{(n)} \equiv_{tt} Z$.

Then $Y \oplus 0^{(n+1)} \geq_{tt} Z \oplus 0^{(n+1)} \geq_{tt} X \geq_{tt} Z' \geq_{tt} Y^{(n+1)}$.

Hence $Y \oplus 0^{(n+1)} \equiv_{tt} Y^{(n+1)} \equiv_{tt} X$, completing the induction.
$\square$

# Coding reals

## Representing the natural numbers

We wish to use a finite set of truth-table degrees to code the information in a real.

We create a set of parameters $\vec{p} \in D_{tt}$ and degrees representing natural numbers $\langle G_n \mid n \in \omega \rangle$.

Each $G_{n+1}$ is the unique truth-table degree satisfying a set of relations involving $\vec{p}$ and $G_n$.

Hence $\vec{p}$ and $G_1$ determine $\langle G_n \mid n \in \omega \rangle$.

# Coding reals (continued)

## Representing a real

To represent a real $S$ we find a degree $q$ such that for every $n \in \omega$ we have that $n \in S$ iff certain relations hold between $G_n$, $q$, and $\vec{p}$.

We note that $\leq_{tt}$ is preserved under automorphisms.

So if $\pi$ is an automorphism and $q$ represents $S$ for $\vec{p}$ and $\langle G_n \mid n \in \omega \rangle$, then $\pi(q)$ still represents $S$ for $\pi(\vec{p})$ and $\langle \pi(G_n) \mid n \in \omega \rangle$.

We use two theorems of Mytilinaios and Slaman to set up this coding system. Let $(X)$ denote the truth-table degrees below $X$.

# Coding reals (continued)

## Theorem (Mytilinaios and Slaman)

*Let $B \geq_{tt} 0'$. Then there exist reals $\vec{p} = \langle E_1, E_2, D_1, D_2, F_1, F_2 \rangle$ with $B \leq_{tt} \vec{p} \leq_{tt} B''$ and $\langle G_n \mid n \in \omega \rangle$ uniformly computable in $B''$ such that:*

1. *For any $G_{n_1}, G_{n_2}, \ldots G_{n_k}$ and $m \neq n_j$ for all $j < k$ we have $(G_{n_1} \oplus \ldots \oplus G_{n_k}) \cap (G_m) = (B)$.*
2. *$D_1 \not\geq_{tt} D_2$ and for any $n \in \omega$ we have $D_1 \oplus G_n \geq_{tt} D_2$.*
3. *For $n$ odd, $(F_1 \oplus G_n) \cap (E_1) = (G_{n+1})$ and For $n$ even, $(F_2 \oplus G_n) \cap (E_2) = (G_{n+1})$.*

## Theorem (Mytilinaios and Slaman)

*Let $B, \vec{p}$, and $\langle G_n \mid n \in \omega \rangle$ satisfy the conditions above.*

*Let $S \subseteq \omega$. Then $S$ is $\Sigma^0_2(\vec{p}) \Leftrightarrow \exists Q[Q \leq_{tt} \vec{p} \wedge \forall n \in \omega$*

*$[n \in S \leftrightarrow \exists X[X \leq_{tt} G_n \wedge X \leq_{tt} Q \wedge D_2 \leq_{tt} X \oplus D_1]]]$.*

# Coding reals (continued)

## Using coding

Given an automorphism $\pi$ and a sufficiently high degree $y$ we can use these theorems to get $\pi(y) \leq_{tt} y''$.

We choose a real $B$ such that $B'' \equiv_{tt} \pi(y)$.

Let $\vec{p}$ be given by the first theorem. We have $B \leq_{tt} \vec{p} \leq_{tt} B''$.

$B''$ is $\Sigma^0_2(\vec{p})$ so we let $q$ code $B''$ for $\vec{p}$.

# Coding reals (conclusion)

## Using coding (continued)

Then $\pi^{-1}(q)$ codes $B''$ for $\pi^{-1}(\vec{p})$.

Hence by the second theorem $B''$ is $\Sigma^0_2(\pi^{-1}(\vec{p}))$.

We have $\vec{p} \leq_{tt} B'' \equiv_{tt} \pi(y)$. Applying $\pi^{-1}$ yields $\pi^{-1}(\vec{p}) \leq_{tt} y$.

Thus $B''$ is $\Sigma^0_2(y)$ so $\pi(y) \equiv_{tt} B'' \leq_{tt} y''$.

# Automorphisms fixed on a cone

## Theorem

*Let $\pi : D_{tt} \to D_{tt}$ be an automorphism. Then there is a degree $b$ such that for all $x \geq_{tt} b$ we have $\pi(x) = x$.*

## Strategy for proof

Let $d$ be a sufficiently high degree and let $x \geq_{tt} d'' \oplus \pi(d'')$.

By strong jump inversion relative to $d$, let $y \geq_{tt} d$ be such that $x \equiv_{tt} y'' \equiv_{tt} y \oplus d''$.

# Automorphisms fixed on a cone (continued)

## Strategy for proof (continued)

As before, we have $\pi(y) \leq_{tt} y''$.

Hence $\pi(x) \equiv_{tt} \pi(y \oplus d'') \equiv_{tt} \pi(y) \oplus \pi(d'') \leq_{tt} y'' \oplus x \leq_{tt} x$.

So $\pi(x) \leq_{tt} x$ and we use symmetry to complete the proof.

# Automorphisms fixed on a cone—Proof

## Proof

Let $d = \pi^{-1}(0''')$ and $e = \pi(0''')$.

Let $b = d'' \oplus \pi(d'') \oplus e'' \oplus \pi^{-1}(e'')$. Let $x \geq_{tt} b$ be arbitrary.

We use the fact that $x \geq_{tt} d'' \oplus \pi(d'')$ to show $\pi(x) \leq_{tt} x$.

By symmetry, $x \geq_{tt} e'' \oplus \pi^{-1}(e'')$ implies $\pi^{-1}(x) \leq_{tt} x$.

Hence $x \leq_{tt} \pi(x)$. Therefore, $\pi(x) = x$ proving the theorem.

# Automorphisms fixed on a cone—Proof (continued)

## Proof (continued)

Since $x \geq_{tt} d''$, by strong jump inversion relative to $d$ there is a $y \geq_{tt} d$ such that $x \equiv_{tt} y'' \equiv_{tt} y \oplus d''$.

We note since $y \geq_{tt} d$ we have $\pi(y) \geq_{tt} \pi(d) = 0'''$.

Hence by jump inversion relative to $0'$, let $Z \geq_{tt} 0'$ be such that $Z'' \equiv_{tt} \pi(y)$.

# Automorphisms fixed on a cone—Proof (continued)

## Proof (continued)

Let $\vec{p} \leq_{tt} Z''$ and $\langle G_n \mid n \in \omega \rangle$ be given by the first coding theorem with base $Z$.

We note $Z''$ is $\Sigma^0_2(Z)$ and $Z \leq_{tt} \vec{p}$ so $Z''$ is $\Sigma^0_2(\vec{p})$.

By the second coding theorem, let $q \leq_{tt} p$ code $Z''$ for $\vec{p}$ and $\langle G_n \mid n \in \omega \rangle$.

# Automorphisms fixed on a cone—Proof (continued)

## Proof (continued)

Since $\leq_{tt}$ is preserved under automorphisms, $\pi^{-1}(q)$ codes $Z''$ for $\pi^{-1}(\vec{p})$ and $\langle \pi^{-1}(G_n) \mid n \in \omega \rangle$.

By the second coding theorem, $Z''$ is $\Sigma^0_2(\pi^{-1}(\vec{p}))$.

Since $\vec{p} \leq_{tt} Z'' \equiv_{tt} \pi(y)$ we have $\pi^{-1}(\vec{p}) \leq_{tt} y$.

Hence $Z''$ is $\Sigma^0_2(y)$, so $Z'' \leq_{tt} y''$.

# Automorphisms fixed on a cone—Proof (conclusion)

## Proof (conclusion)

Since $y'' \equiv_{tt} x$ and $\pi(y) \equiv_{tt} Z'' \leq_{tt} y''$, we conclude $\pi(y) \leq_{tt} x$.

Since $x \geq_{tt} b$ we have $\pi(d'') \leq_{tt} x$. Thus $\pi(y) \oplus \pi(d'') \leq_{tt} x$.

This implies $y \oplus d'' \leq_{tt} \pi^{-1}(x)$.

Since $y \oplus d'' \equiv_{tt} x$ we have $x \leq_{tt} \pi^{-1}(x)$.

Therefore, $\pi(x) \leq_{tt} x$, proving the theorem. $\square$

# Remarks

## Calculating the base of the cone

We note that if $\pi(0''') = 0'''$ and $\pi(0^{(5)}) = 0^{(5)}$ then the base of the cone is $0^{(5)}$.

In particular, for the structure of the truth-table degrees with jump, $(D_{tt}, \leq_{tt}, ')$, the base of the cone is $0^{(5)}$.

This is one jump weaker than Kjos-Hanssen obtained by working with the structure $(D_{tt}, \leq_{tt}, ')$ directly.

# More applications to automorphisms

## More results

There are several related results on automorphisms of $D$ which can be adapted to automorphisms of $D_{tt}$.

For example:

Let $b$ be the base of the cone where $\pi = \mathrm{id}$. Let $I$ be an ideal in $D_{tt}$ with $b \in I$.

Then the restriction $\pi \upharpoonright I$ is an automorphism of $I$.

To see this, let $x \in I$. Then $x \oplus b \in I$ and $b \leq_{tt} x \oplus b$.

Hence $\pi(x) \leq_{tt} \pi(x \oplus b) = x \oplus b$ so $\pi(x) \in I$.

This can also be done with $\pi^{-1}$ to complete the proof.

# Conclusion

## Further progress

Can we prove truth-table analogues of some of the results of Slaman and Woodin on automorphisms of the Turing degrees?

Main obstacle is finding a way to code a countable antichain in $D_{tt}$ using a finite set of parameters.

# Conclusion (continued)

## Coding antichains

Let $A = \langle A_n \in D_{tt} \mid n \in \omega \rangle$ be a countable antichain.

We wish to find a formula $\psi$ in the language $(D_{tt}, \leq_{tt})$ and a finite set of parameters $\vec{p}$ such that:

$x \in A \Leftrightarrow \psi(\vec{p}, x)$.

The parameters $\vec{p}$ must be arithmetic in $A$

Preferably, $\vec{p}$ should be $\Sigma^0_n(A)$ for some low $n$.

# Conclusion (continued)

## Results of coding antichains

Results which seem to follow from coding countable antichains in the truth-table degrees:

The statement "There is a nontrivial automorphism of the truth-table degrees" is absolute between well-founded models of ZFC.

It is a necessary step in applying the methods of Slaman and Woodin to automorphisms of $D_{tt}$.

# Conclusion (continued)

## How to code antichains?

Slaman and Woodin use genericity to code countable antichains in the Turing degrees.

The proof relies heavily on the fact that every Turing degree contains a real which is computable in each of its infinite subsets.

A corresponding fact is false for the truth-table degrees.

# Conclusion (end)

## Other open questions

Let $X \geq_{tt} 0'$ be a $\Sigma_2^0$ real. Is there a c.e. real $Y$ such that $Y' \equiv_{tt} X$?

What if we weaken the requirement that $Y$ is c.e. to $Y \leq_{tt} 0'$ or to $Y$ is $\Delta_2^0$ ?

Is $\leq_T$ definable in $(D_{tt}, \leq_{tt})$ ?