Reals *n*-generic relative to a perfect tree

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May 13, 2006

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Introduction

Relative Properties

One area of study in Recursion Theory is which reals X hold some property relative to another real Y.

- Examples: For which reals *X* does there exist a *Y* such that
 - *X* is properly r.e. (or Σ_n) in *Y*
 - ► X is *n*-generic in Y
 - ► X is *n*-random in Y

For these questions, we work at the level of reals, not of Turing degrees.

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Motivation

Relative Properties (continued)

These investigations are interesting when reals with traits quite different from some property, still have this property in some context.

For example, a real X of minimal degree can be (properly) r.e. relative to some Y or "*n*-generic" relative to some Y.

A property I have studied previously is relative definability and classifying which reals are relatively r.e.

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Motivation (continued)

Relative Genericity

On the other side of the scale is the question: When is a real generic in some relative context?

Here the best framework is studying reals that are generic when viewed as a path through some perfect tree, instead of all of $2^{<\omega}$.

We find some reals with high information content, such at the theory of second order arithmetic, can appear generic. Other reals with much lower levels of complexity, such as \emptyset' , can not.

Related Work

Connections

In addition to relative genericity, people have also studied:

- Reals random relative to a continuous measure (Reimann and Slaman)
- Relatively hyperimmune-free reals

These three relative properties hold for very different sets of reals.

Yet, all display the same limiting behavior given for relative genericity in the two main results for this talk.

On the other hand, the set of relatively r.e. reals do not have this limiting behavior.

Definitions

Definition

A real *X* is *n*-generic relative to a perfect tree *T* if *X* is a path through *T* and for all $\Sigma_n(T)$ sets *S*, there is a *k* such that either $X|k \in S$ or $\sigma \notin S$ for every $\sigma \in T$ extending X|k.

Definition

A real *X* is *n*-generic relative to some perfect tree if there exists a perfect tree *T* such that *X* is *n*-generic relative to *T*.

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Notation

Definition

We will use GPT_n to denote the set of reals *n*-generic relative to some perfect tree and $\neg GPT_n$ to denote its complement.

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Definition

 $X \equiv_{T,A} Y$ will be used to mean $X \oplus A \equiv_T Y \oplus A$.

Definition

We let δ be the least ordinal such that $\sup_{\beta < \delta} (\beta \text{th admissible}) = \delta.$

Main Results

Theorem For all $n \in \omega$, the set $\neg GPT_n$ is countable.

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Main Results

Theorem For all $n \in \omega$, the set $\neg GPT_n$ is countable.

Theorem *For all* $n \ge 5$ *and* $\alpha < \delta$ *, the iterated hyperjump* $\mathcal{O}^{(\alpha)} \in \neg GPT_n$ *.*

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 $\neg GPT_n$ is countable — Prior Results

Theorem (Martin)

Let B be a Borel set of reals such that for every $X \in 2^{\omega}$ there exists a $Y >_T X$ such that $Y \in B$. Then there is a Turing Degree **b** such that for every degree $d \ge_T b$ there is a set Z in d with $Z \in B$.

Let $B \subseteq 2^{\omega} \times 2^{\omega}$ denote a set of reals where the first real holds some property relative to the second.

Let $B^Z = \{X \mid (X, Z) \in B\}$ and *B* abbreviate B^{\emptyset} .

 $\neg GPT_n$ is countable — Prior Results (continued)

Theorem (Reimann and Slaman)

Let B be a Borel set as above such that B^G is cofinal in the Turing degrees for all G. Then for all but countably many reals X, there exists reals Y and G such that $X \equiv_{T,G} Y$ and $Y \in B^G$.

The countable set used is $L_{\beta} \cap 2^{\omega}$ where β is least such that

 $L_{\beta} \models \exists$ uncountably many cardinals

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 $\neg GPT_n$ is countable — Requirements

Need:

To prove $\neg GPT_n$ is countable, we need to find a set *B* such that:

- 1. *B* is Borel.
- 2. B^G is cofinal in the Turing degrees for all G.
- 3. For all *X*, *Y*, *G* \in 2^{ω} we have

$$(X \equiv_{T,G} Y \text{ and } Y \in B^G) \Rightarrow X \in GPT_n$$

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 $\neg GPT_n$ is countable — Main Lemma

Solution:

Let *B* be the set of reals of Turing degree $X \oplus A$ for any *X*, *A* such that *X* is (n + 1)-generic (*A*).

Lemma

Let $n \ge 1$, *A* be a set, *X* be (n + 1)-generic (*A*) and $X \equiv_{T,A} Y$. Then *Y* is *n*-generic relative to a perfect tree.

 $\neg GPT_n$ is countable — Proof of Theorem

Given the Lemma: Fix $n \in \omega$ and define *B* by

 $B = \{(X,G) | \exists A \exists H [X \equiv_{T,G} H \oplus A \text{ and } H \text{ is } (n+1) \text{-generic } (A \oplus G) \}$

Consider *X*, *Y*, *G* such that $X \equiv_{T,G} Y$ and $Y \in B^G$.

This means there exist *A*, *H* such that $A \oplus H \equiv_{T,G} Y$ and *H* is (n + 1)-generic $(A \oplus G)$.

Then $Y \equiv_{T,A \oplus G} H$ so $X \equiv_{T,A \oplus G} H$. By the lemma, $X \in GPT_n$

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 $\neg GPT_n$ is countable — Proof of Lemma

To prove the Lemma:

Let Ψ : $X \to Y$ and Φ : $Y \to X$ be *A*-recursive Turing reductions that witness $X \equiv_{T,A} Y$.

Since *X* is at least 2-generic (*A*), choose $p \in X$ such that

 $p \Vdash \Phi \circ \Psi = \mathrm{id} \land \Psi \mathrm{total}$

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Let *T* be the tree of possible initial segments of *Y*, $T = \{ \sigma \mid \exists q \supseteq p[\sigma \subseteq \Psi(q)] \}.$ $\neg GPT_n$ is countable — Proof of Lemma (continued)

Proof (cont.):

We wish to show Υ is *n*-generic relative to *T*. (We lose a quantifier since *T* is $\Sigma_1(A)$, not $\Delta_1(A)$).

Let *S* be an arbitrary $\Sigma_n(T)$ set. Consider the pullback

$$\Psi^{-1}(S) = \{q \mid \exists r [\Psi(q) \supseteq r \land r \in S\}$$

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We apply the genericity of *X* for the pullback to get the genericity of *Y* for *S*.

 $\neg GPT_n$ is countable — Proof of Lemma (continued)

Proof (cont.):

T is $\Sigma_1(A)$ so *S* is $\Sigma_{n+1}(A)$ and $\Psi^{-1}(S)$ is $\Sigma_{n+1}(A)$. Since *X* is (n + 1)-generic (*A*), we have two possible cases.

Case 1: $\exists n[X|n \in \Psi^{-1}(S)]$. We then let *m* be such that $Y|m \subseteq \Psi(X|n)$ and $Y|m \in S$.

 $\neg GPT_n$ is countable — Proof of Lemma (conclusion)

Proof (cont.):

Case 2: $\exists n (\forall q \supseteq X | n) [q \notin \Psi^{-1}(S)].$

Let *m* be such that $\Phi(Y|m) \supseteq X|n$.

We now use Y|m, (instead of $\Psi(X|n)$), to witness genericity of Y for S. \Box

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 $\mathcal{O} \in \neg GPT_2$ — Preliminaries

Lemma $\mathcal{O} \in \neg GPT_2$.

Definitions

For $e \in \omega$, let U_e denote the *e*th recursive tree in $\omega^{<\omega}$.

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We view \mathcal{O} as $\{e \mid U_e \text{ is well founded }\}$.

 $\mathcal{O} \in \neg GPT_2$ — Proof

Strategy for proof

- ▶ We assume towards a contradiction that *O* is 2-generic relative to the perfect tree *T*.
- We note O has the property that the well-foundedness of subtrees cannot contradict the decision made for the parent tree.
- We find a Σ₂ set S of strings which witness a failure of this property.

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$\mathcal{O} \in \neg GPT_2$ — Proof (continued)

Strategy for proof (continued)

- ▶ For strings not in *S*, statements of ill-foundedness have possible extensions in *T* witnessing the ill-foundedness of subtrees with arbitrarily long root length
- ▶ We note that since *O* is 2-generic relative to *T* then *T* has a perfect subtree with no branches in *S*.
- We claim that for such a well behaved *T*, we get \mathcal{O} is $\Pi_1(T)$, for a contradiction.
- ► To accomplish the claim, we show a tree is ill-founded iff a string in the well behaved part of *T* says it is ill-founded.

 $\mathcal{O}^{(\alpha)} \in \neg GPT_5$ — Finite levels

Lemma

Let $X \ge_T O$ be 2-generic relative to the perfect tree T. Then $T \ge_T O$.

Proof (sketch)

Apply the previous lemma to the column of X computing O.

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Corollary For all $n \in \omega$, $\mathcal{O}^{(n)} \in \neg GPT_2$

Corollary Π_1^1 -*CA does not prove* " \neg *GPT*₂ *is countable.*"

 $\mathcal{O}^{(\alpha)} \in \neg GPT_5$ — Transfinite levels

Lemma (Slaman)

Let *A* be a set and λ a recursive limit ordinal. Suppose that for all $\beta < \lambda$ we have $\mathcal{O}^{(\beta)} \leq_T A$. Then $\mathcal{O}^{(\lambda)}$ is $\Sigma_5(A)$.

Theorem For all $n \ge 5$ and $\alpha < \delta$, the iterated hyperjump $\mathcal{O}^{(\alpha)} \in \neg GPT_n$.

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Other Results

Lemma Let $X \in 2^{\omega}$ be ranked. Then $X \in \neg GPT_1$.

Lemma (Slaman) Let $n \in \omega$ and X have n-REA degree. Then $X \in \neg GPT_1$.

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Lemma *There exists a 1-generic* ω *-r.e. real.*