Degrees that are not Degrees of Categoricity

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March 26, 2011

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Computable structures

Definition

A structure (coded as a subset of ω) is a computable structure if its domain and atomic diagram are computable.

Without loss of generality, we assume all computable structures have domain ω.

Notation

We denote the n-th computable structure under some effective listing by $\mathcal{A}_n$.

Computably categorical structures

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Let $\mathcal{A}$ be a computable structure. We say that $\mathcal{A}$ is computably categorical if for every computable structure $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism $f : \mathcal{A} \to \mathcal{B}$.

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Example

Given two computable copies of the dense linear orders without endpoints (DLO) we can find a computable isomorphism between them.

Therefore they are computably categorical structures.

Relatively computably categorical structures

Definition

Let $\mathcal{A}$ be a computable structure and $\mathbf{x}$ a Turing degree. We say that $\mathcal{A}$ is **x**-computably categorical if for every computable structure $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ with $f \leq_T \mathbf{x}$.

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Example

The standard ordering on $\mathbb{N}$ is $\mathbf{0}'$-computably categorical.

To build an isomorphism to a computable copy, we use $\mathbf{0}'$ to determine how many predecessors each element has.

Definition

$\mathrm{CatSpec}(\mathcal{A}) = \{\mathbf{x} \mid \mathcal{A} \text{ is } \mathbf{x}\text{-computably categorical}\}$

Degrees of categoricity

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Definition (Fokina, Kalimullin, and Miller)

A Turing degree $\mathbf{x}$ is a degree of categoricity if there is a computable structure $\mathcal{A}$ such that $\mathbf{x} \in \mathrm{CatSpec}(\mathcal{A})$ and for all $\mathbf{y} \in \mathrm{CatSpec}(\mathcal{A})$ we have $\mathbf{x} \leq_T \mathbf{y}$.

Degrees of categoricity are sometimes called categorically definable degrees.

Degrees of categoricity (continued)

Summary

$\mathcal{A}$ witnesses $\mathbf{x}$ is a degree of categoricity if $\mathbf{x}$ is the least degree that can compute isomorphisms between $\mathcal{A}$ and any computable structure isomorphic to it.

Example

For example, computable copies of the DLO witness that $\mathbf{0}$ is a degree of categoricity.

Strong degrees of categoricity

Definition

A Turing degree $\mathbf{x}$ is a strong degree of categoricity if there is a computable structure $\mathcal{A}$ with computable copies $\mathcal{B}$ and $\mathcal{M}$ such that $\mathcal{A}$ is $\mathbf{x}$-computably categorical, and for every isomorphism $f : \mathcal{B} \to \mathcal{M}$ we have $\mathbf{x} \leq_T f$.

Remark

Strong degrees of categoricity are degrees of categoricity.

Known results (positive)

Fokina, Kalimullin, and Miller developed the basic method for showing degrees are degrees of categoricity.

Theorem (Fokina, Kalimullin, and Miller)

Let $\mathbf{x}$ *be a d.c.e. degree. Then* $\mathbf{x}$ *is a [strong] degree of categoricity.*

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Theorem (Fokina, Kalimullin, and Miller)

Let $\mathbf{x}$ be a d.c.e. degree. Then $\mathbf{x}$ is a [strong] degree of categoricity.

This result can be relativized to finite and transfinite jumps.

Theorem (Fokina, Kalimullin, and Miller)

Let $n \in \omega$ and let $\mathbf{x}$ be d.c.e.($\emptyset^{(n)}$) with $\mathbf{x} \geq_T \emptyset^{(n)}$. Then $\mathbf{x}$ is a [strong] degree of categoricity.

Theorem (Csima, Franklin, and Shore)

Let $\alpha < \omega_1^{CK}$ and let $\mathbf{x}$ be d.c.e.($\emptyset^{(\alpha)}$) with $\mathbf{x} \geq_T \emptyset^{(\alpha)}$. Then $\mathbf{x}$ is a [strong] degree of categoricity.

Known results (negative)

It is easy to see that there are at most countably many degrees of categoricity.

It has been shown that all degrees of categoricity are hyperarithmetical.

Theorem (Fokina, Kalimullin, and Miller)

If $\mathbf{x} \notin HYP$, then $\mathbf{x}$ is not a strong degree of categoricity.

Theorem (Csima, Franklin, and Shore)

If $\mathbf{x} \notin HYP$, then $\mathbf{x}$ is not a degree of categoricity.

Warm up proposition

In this talk we will show several more negative results. We start by considering a straight-forward example.

Proposition (Anderson and Csima)

There is a degree $\mathbf{x} \leq_T \mathbf{0}''$ *that is not a degree of categoricity.*

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There is a degree $\mathbf{x} \leq_T \mathbf{0}''$ *that is not a degree of categoricity.*

Ideas for proof

- We construct a noncomputable X by finite extensions using a $\emptyset''$ oracle.
- We build X so that for any computable structure $\mathcal{A}_m$ we have $\mathrm{Deg}(X) \in \mathrm{CatSpec}(\mathcal{A}_m) \Rightarrow \mathbf{0} \in \mathrm{CatSpec}(\mathcal{A}_m)$.

Warm up proposition (continued)

Ideas for proof (continued)

- For every (l, m, k) we want to satisfy:
  Either Φ_l^X is not an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, or there is a computable isomorphism.

Warm up proposition (continued)

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- For every (l, m, k) we want to satisfy:
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- Given a string σ we wish to determine if there is a $\tau \supseteq \sigma$ such that Φ_l^τ cannot be extended to an isomorphism.

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- We ask $\emptyset'$: Is there a $\tau \supseteq \sigma$ such that Φ_l^τ is seen not to be an injective homomorphism?

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- We ask $\emptyset''$: Is there a $\tau \supseteq \sigma$ and a $d \in \omega$ such that for every $\gamma \supseteq \tau$ we have d is not in the domain or range of Φ_l^γ?

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- Yes: extend to τ. No: there is a computable isomorphism.

2-generic relative to some perfect tree

We wish to generalize this proof to come up with a negative result on a broad class of sets.

Definition

A set G is *n-generic* if for every Σ_n subset S of $2^{<\omega}$ there is an l such that either $G \upharpoonright l \in S$ or for all $\tau \supseteq G \upharpoonright l$ we have $\tau \notin S$.

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Definition

A set G is n-generic relative to the perfect tree T if G is a path through T and for every $\Sigma_n(T)$ subset S of T, there is an l such that either $G \upharpoonright l \in S$ or for all $\tau \supseteq G \upharpoonright l$ with $\tau \in T$ we have $\tau \notin S$.

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Definition

A set G is n-generic relative to some perfect tree if there exists a perfect tree T such that G is n-generic relative to T.

2-generic relative to some perfect tree (continued)

We can now use this to limit degrees of categoricity to a small, easily defined class.

Theorem (Anderson)

For every n, there are only countably many sets that are not n-generic relative to any perfect tree.

2-generic relative to some perfect tree (continued)

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Theorem (Anderson)

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Generalizing the methods used to construct a degree below $\mathbf{0}''$ we can prove:

Theorem (Anderson and Csima)

Let G be 2-generic relative to some perfect tree and $\mathbf{g} = Deg(G)$. Then $\mathbf{g}$ is not a degree of categoricity.

The theorem allows us to find a degree that is not a degree of categoricity between any set and its double jump.

Corollary

Let X and A be sets such that X is 2-generic (A). Then $\mathbf{x} \oplus \mathbf{a}$ is not a degree of categoricity.

Corollary

For every $\mathbf{x}$ there is a $\mathbf{y}$ such that $\mathbf{x} \leq_T \mathbf{y} \leq_T \mathbf{x}''$ and $\mathbf{y}$ is not a degree of categoricity.

Hyperimmune-free

We can also exclude degrees of categoricity from another class.

Definition

A degree $\mathbf{x}$ is hyperimmune-free if for every function $f \leq_T \mathbf{x}$ there is a computable function g which dominates f.

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We notice that all known degrees of categoricity are between jumps and hence hyperimmune.

Theorem (Anderson and Csima)

Let $\mathbf{x}$ be a noncomputable hyperimmune-free degree. Then $\mathbf{x}$ is not a degree of categoricity.

Σ_2 Degree

There are no hyperimmune-free degrees or degrees of sets 2-generic relative to some perfect tree that are Σ_2.

However, we can construct a Σ_2 set whose degree is not a degree of categoricity directly.

Theorem (Anderson and Csima)

There is a Σ_2 degree that is not a degree of categoricity.

Σ_2 Degree (continued)

Ideas for proof

- We construct X to be c.e. in a $\emptyset'$ oracle.

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- Unlike our earlier construction, we can no longer ask $\emptyset''$ oracle questions.
- We weaken the requirement that $\mathbf{x} \in \mathrm{CatSpec}(\mathcal{A}_m) \Rightarrow \mathbf{0} \in \mathrm{CatSpec}(\mathcal{A}_m)$.
- Instead, for each $m \in \omega$ we construct a $Y_m \not\geq_T X$ such that for all k, if X computes an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$ then so does Y_m.

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- Unlike our earlier construction, we can no longer ask $\emptyset''$ oracle questions.
- We weaken the requirement that $\mathbf{x} \in \text{CatSpec}(\mathcal{A}_m) \Rightarrow \mathbf{0} \in \text{CatSpec}(\mathcal{A}_m)$.
- Instead, for each $m \in \omega$ we construct a $Y_m \not\geq_T X$ such that for all k, if X computes an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$ then so does Y_m.
- Each Y_m witnesses $\mathbf{x}$ is not the least degree in $\text{CatSpec}(\mathcal{A}_m)$.

Σ_2 Degree (continued)

Ideas for proof (continued)

- We split each Y_m into columns, $Y_m^{[l,k]}$.
- We maintain $Y_m^{[l,k]}(t) = 0 \Rightarrow X(t) = 0$ for all t.

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- If we appear unable to block Φ_l^X from becoming an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, we will try to make $f = \Phi_l^{Y_m^{[l,k]}}$ an isomorphism.

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- We build X by finite extensions except at special stages called slides.

Σ_2 Degree (continued)

Ideas for proof (continued)

- Given σ we ask $\emptyset'$ if there is a $\tau \supseteq \sigma$ such that Φ_l^τ is not a partial injective homomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$.
- At this point we have [roughly speaking] $X \upharpoonright \sigma = Y_m^{[l,k]} \upharpoonright \sigma$.

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- If yes, we extend to τ and are done for (l, m, k).
- If no, then for all $\gamma \supseteq \sigma$ we have Φ_l^γ is a partial injective homomorphism.

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- At this point we have [roughly speaking] $X \upharpoonright \sigma = Y_m^{[l,k]} \upharpoonright \sigma$.
- If yes, we extend to τ and are done for (l, m, k).
- If no, then for all $\gamma \supseteq \sigma$ we have Φ_l^γ is a partial injective homomorphism.
- We attempt to build $Y_m^{[l,k]} \supseteq \sigma$ by finite extensions to ensure every $d \in \omega$ is in the domain and range of $f = \Phi_l^{Y_m^{[l,k]}}$.

Σ_2 Degree (continued)

Ideas for proof (conclusion)

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- In this case we perform a slide. We change $X(t)$ from 0 to 1 for all t where X differs from $Y_m^{[l,k]}$.
- We now have $X = Y_m^{[l,k]}$ and since Φ_l^X cannot be made into an isomorphism, we are done for (l, m, k).

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- We now have $X = Y_m^{[l,k]}$ and since Φ_l^X cannot be made into an isomorphism, we are done for (l, m, k).
- Many weaker priorities are injured, but a finite injury construction is possible.

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There is still a lot of open ground in determining how simple a degree can be without being a degree of categoricity.

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3. Is there a degree of categoricity which is not a strong degree of categoricity?

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Thank you.