Section 5.2

Ring Homomorphisms

RING HOMOMORPHISM

5.2.1 Definition. Let R and S be commutative rings. A function $\phi : R \to S$ is called a ***ring homomorphism*** if

(i) $\phi(a+b) = \phi(a) + \phi(b)$, for all $a, b \in R$;

(ii) $\phi(ab) = \phi(a)\phi(b)$, for all $a, b \in R$; and

(iii) $\phi(1) = 1$.

A ring homomorphism that is one-to-one and onto is called a ***ring isomorphism***. If there is a ring isomorphism from R onto S, we say that R is ***isomorphic*** to S, and write $R \cong S$.

A ring isomorphism from the commutative R onto itself is called and ***automorphism*** of R.

INVERSES AND COMPOSITION OF RING ISOMORPHISMS

5.2.2 Proposition.

(a) The inverse of a ring isomorphism is a ring isomorphism.

(b) The composite of two ring isomorhisms is a ring isomorphism.

EXAMPLES OF HOMOMORPHISMS

- **Natural Projection.** The mapping $\pi : \mathbb{Z} \to \mathbb{Z}_n$ given by $\pi(x) = [x]_n$, for all $x \in \mathbb{Z}$, is a ring homomorphism that is onto but not one-to-one.
- **Natural Inclusion.** Let R be a commutative ring, and define $\iota : R \to R[x]$ by $\iota(a) = a$, for all $a \in R$. That is, $\iota(a)$ is the constant polynomial a. This is a ring homomorphism that is one-to-one but not onto.

EVALUATION MAPPING

Let F be a subfield of the field E. For any element $u \in E$, we can define a function $\phi_u : F[x] \to E$ by letting $\phi_u(f(x)) = f(u)$, for each $f(x) \in F[x]$. Then the requirements for a ring homomorphism are satisfied. Since the polynomials in $F[x]$ are evaluated at u, the homomorphism ϕ_u is called an ***evaluation mapping***.

BASIC PROPERTIES OF RING HOMOMORPHISMS

5.2.3 Proposition. Let $\phi : R \to S$ be a ring homomorphism. Then

(a) $\phi(0) = 0$;

(b) $\phi(-a) = -\phi(a)$ for all $a \in R$;

(c) $\phi(R)$ is a subring of S.

KERNEL OF A RING HOMOMORPHISM

5.2.4 Definition. Let $\phi : R \to S$ be a ring homomorphism. The set

$$\{a \in R \mid \phi(a) = 0\}$$

is called the ***kernel*** of ϕ, denoted by $\ker(\phi)$.

PROPERTIES OF THE KERNEL

5.2.5 Proposition. Let $\phi : R \to S$ be a ring homomorphism.

(a) If $a, b \in \ker(\phi)$ and $r \in R$, then $a + b$, $a - b$, and ra belong to $\ker(\phi)$.

(b) The homomorphism ϕ is an isomorphism if and only if $\ker(\phi) = \{0\}$ and $\phi(R) = S$.

FUNDAMENTAL HOMOMORPHISM THEOREM FOR RINGS

5.2.6 Theorem. Let $\phi : R \to S$ be a ring homomorphism. Then $R/\ker(\phi) \cong \phi(R)$.

GENERALIZED SETTING FOR THE EVALUATION MAPPING

5.2.7 Proposition. Let R and S be commutative rings, let $\theta : R \to S$ be a ring homomorphism, and let s be any element of S. Then there exists a unique ring homomorphism $\hat{\theta}_s : R[x] \to S$ such that $\hat{\theta}_s(r) = \theta(r)$ for all $r \in R$, and $\hat{\theta}_s(x) = s$.

ROOTS OF POLYNOMIALS

Let R be a subring of the ring S, and let $\theta : R \to S$ be the inclusion mapping. If $s \in$ S, then the ring homomorphism $\hat{\theta}_s : R[x] \to S$ defined by Proposition 5.2.7 should be thought of as an evaluation mapping, since $\hat{\theta}_s\big(f(x)\big) = f(s)$, for any polynomial $f(x) \in R[x]$. If $f(s) = 0$, then we say that s is a ***root*** of the polynomial $f(x)$.

COMPONENT-WISE ADDITION AND MULTIPLICATION

5.2.8 Proposition. Let $R_1, R_2, \dots, R_n$ be commutative rings. The set of n-tuples $(a_1, a_2, \dots, a_n)$ such that $a_i \in R_i$ for each i is a commutative ring under the following addition and multiplication:

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

$$(a_1, a_2, \dots, a_n) \cdot (b_1, b_2, \dots, b_n) = (a_1 b_1, a_2 b_2, \dots, a_n b_n)$$

For n-tuples $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$.

DIRECT SUM

5.2.9 Definition: Let $R_1, R_2, \dots, R_n$ be commutative rings. The set of n-tuples $(a_1, a_2, \dots, a_n)$ such that $a_i \in R_i$ for each i, under the operations of component-wise addition and multiplication is called the ***direct sum*** of the commutative rings $R_1, R_2, \dots, R_n$, and is denoted by

$$R_1 \oplus R_2 \oplus \cdots \oplus R_n .$$

CHARACTERISTIC OF A RING

5.2.10 Defintion. Let R be a commutative ring. The smallest positive integer n such that $n \cdot 1 = 0$ is called the ***characteristic*** of R, denoted by $\mathrm{char}(R)$.

If no such positive integer exists, then R is said to have ***characteristic zero***.

INTEGRAL DOMAINS AND CHARACTERISTIC

5.2.11 Proposition. An integral domain has characteristic 0 or p, for some prime number p.