Section 9.3

The Three-Term Taylor Method

TAYLOR SERIES

Recall that the Taylor series expansion of a function y centered at $x = a$ is given by

$$y(x) = y(a) + y'(a)\frac{(x-a)}{1!} + y''(a)\frac{(x-a)^2}{2!} + \cdots$$

and this converges for $|x - a| < R$. Notice that if we set $a = x_n$ and $x = x_n + h$, then the formula above becomes

$$y(x_n + h) = y(x_n) + y'(x_n)h + y''(x_n)\frac{h^2}{2} + \cdots$$

EULER'S FORMULA REVISITED

Assume $y(x)$ is a solution of the first-order differential equation $y' = f(x, y)$. If we truncate the Taylor series after the first two terms, we get

$$\begin{aligned} y(x_n + h) &\approx y(x_n) + y'(x_n)h \\ &\approx y(x_n) + f(x_n, y(x_n))h \end{aligned}$$

Notice that we can obtain Euler's formula

$$y_{n+1} = y_n + hf(x_n, y_n)$$

by replacing $y(x_n + h)$ and $y(x_n)$ by their approximations y_{n+1} and y_n, respectively.

TAYLOR'S METHOD

By retaining the first three terms in the Taylor series, we can write

$$y(x_n + h) \approx y(x_n) + y'(x_n)h + y''(x_n)\frac{h^2}{2}$$

After the replacements noted for Euler's formula, we obtain

$$y_{n+1} = y_n + y_n'h + y_n''\frac{h^2}{2}$$

NOTE: The second derivative y'' is obtained by differentiating $y' = f(x, y)$.