Section 7.4

Transforms of Derivatives, Integrals, and Periodic Functions

TRANSFORM OF A DERIVATIVE

Theorem: If $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathcal{L}\{f(t)\}$.

THE CONVOLUTION

If the functions f and g are piecewise continuous on $[0, \infty)$, then the **convolution** of f and g, denoted by $f * g$, is given by the integral.

$$f * g = \int_0^t f(\tau) g(t - \tau) d\tau$$

Exercise 29 will show that $f * g = g * f$.

THE CONVOLUTION THEOREM

Theorem: Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\, \mathcal{L}\{g(t)\} = F(s)G(s).$$

NOTE: If $g(t) = 1$ and $G(s) = \frac{1}{s}$, then the convolution theorem gives

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}.$$

INVERSE OF THE CONVOLUTION THEOREM

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$$

TRANSFORM OF A PERIODIC FUNCTION

Theorem: Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order. If $f(t)$ is periodic with period T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt\,.$$