Section 7.3

Translation Theorems and Derivatives of a Transform

THE FIRST TRANSLATION THEOREM

Theorem: If a is any real number, then

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s\to s-a} = F(s-a)$$

where $F(s) = \mathcal{L}\{f(t)\}$.

NOTE: This theorem is sometimes called the **First Shifting Theorem**.

INVERSE FORM OF THE FIRST TRANSLATION THEOREM

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\left\{F(s)\Big|_{s\to s-a}\right\} = e^{at}f(t)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

THE UNIT STEP FUNCTION

The **unit step function** $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

THE SECOND TRANSLATION THEOREM

Theorem: If a is a positive constant, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s),$$

where $F(s) = \mathcal{L}\{f(t)\}$.

NOTE: This theorem is also called the **Second Shifting Theorem**.

LAPLACE TRANSFORM OF THE UNIT STEP FUNCTION

Using the Second Translation Theorem and identifying $f(t) = 1$, we find $f(t-a) = 1$ and $F(s) = \mathcal{L}\{1\} = \frac{1}{s}$. Therefore, the Laplace transform of the unit step function is

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$$

INVERSE OF THE SECOND TRANSLATION THEOREM

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

where $a > 0$ and $f(t) = \mathcal{L}^{-1}\{F(s)\}$

DERIVATIVES OF TRANSFORMS

Theorem: For $n = 1,2,3, \ldots$,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$

where $F(s) = \mathcal{L}\{f(t)\}$.