Section 7.1

Laplace Transform

INTEGRAL TRANSFORM

A definite integral of the form $\int_a^b K(s,t)f(t)dt$ transforms a function $f(t)$ into a function of s. This is called an **integral transform**.

We are particular interested in this transform where the interval of integration is $[0,\infty)$. That is, if $f(t)$ is defined for $t \geq 0$, then the improper integral is defined by means of a limit.

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b\to\infty}\int_0^b K(s,t)f(t)dt$$

If the limit exists, the integral converges. Otherwise, it diverges.

LAPLACE TRANSFORM

Definition: Let $f(t)$ be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$$

is said to be a **Laplace Transform** of f provided the integral converges.

NOTE: We use the corresponding uppercase letter to denote the Laplace transform; *e.g.*,

$$\mathcal{L}\{f(t)\} = F(s).$$

LINEARITY OF THE LAPLACE TRANSFORM

The Laplace transform is a **linear transform**, or **linear operator**, since

$$\begin{aligned}\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \alpha\,\mathcal{L}\{f(t)\} + \beta\,\mathcal{L}\{g(t)\} \\ &= \alpha\,F(s) + \beta\,G(s)\end{aligned}$$

EXPONENTIAL ORDER

Definition: A function f is said to be of exponential order if there exist numbers $c > 0$, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for $t > T$.

NOTE: This just says that the graph of $f(t)$ does not grow faster than the exponential function Me^{ct}.

SUFFICIENT CONDITIONS FOR EXISTENCE OF LAPALCE TRANSFORM

Let $f(t)$ be piecewise continuous on the interval $[0,\infty)$ and of exponential order for $t > T$; then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

NOTE: These conditions are *not* necessary for the existence of a Laplace transform. The function $f(t) = t^{-1/2}$ is not piecewise continuous on $[0,\infty)$, but its Laplace transform exists.

LAPLACE TRANSFORMS OF SOME BASIC FUNCTIONS

1. $\mathcal{L}\{1\} = \frac{1}{s}$
2. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, n = 1, 2, 3, \ldots$
3. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
4. $\mathcal{L}(t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}, n = 1, 2, 3, \ldots$
5. $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$
6. $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$