Section 6.3

Solutions about Ordinary Points

ORDINARY AND SINGULAR POINTS

Definition: Consider a homogeneous second order differential equation in standard form

$$y'' + P(x)y' + Q(x)y = 0.$$

A point x_0 is said to be an **ordinary point** of the differential equation if both $P(x)$ and $Q(x)$ are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

RECALL: A function is analytic at the point x_0 if it can be represented by a power series in $(x - x_0)$ with $R > 0$.

ORDINARY AND SINGULAR POINTS OF DEs WITH POLYNOMIAL COEFFICIENTS

Given the homogeneous second order equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Where $a_2(x)$, $a_1(x)$, and $a_0(x)$ are polynomials with ***no common factors***, a point $x = x_0$ is

(i) an ordinary point if $a_2(x_0) \neq 0$ or

(ii) a singular point i $a_2(x_0) = 0$.

EXISTENCE OF A POWER SERIES SOLUTION

Theorem: If $x = x_0$ is an ordinary point of the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, we can always find two linearly independent solutions in the form of power series centered at x_0:

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

A series solution converges at least for $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point (real or complex).

COMMENTS

1. For the sake of simplicity, we assume an ordinary point is always located at $x = 0$, since, if not, the substitution $t = x - x_0$ translates the value $x = x_0$ to $t = 0$.
2. The distance from the ordinary point $x = 0$ to a complex singular point $x = a + bi$ is the **modulus** (**magnitude**) of the complex number. The modulus, $|x|$, of $x = a + bi$ is defined to be

$$|x| = \sqrt{a^2 + b^2}$$

NONPOLYNOMIAL COEFFICIENTS

To deal with homogeneous second order equations with nonpolynomial coefficients, we expand the coefficients as power series centered at the ordinary point $x = 0$.