Section 6.1

Cauchy-Euler Equation

THE CAUCHY-EULER EQUATION

Any linear differential equation of the from

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

where $a_n, \ldots, a_0$ are constants, is said to be a **Cauchy-Euler equation**, or **equidimensional equation**.

NOTE: The powers of x match the order of the derivative.

HOMOGENEOUS 2ND-ORDER CAUCHY-EULER EQUATION

We will confine our attention to solving the homogeneous second-order equation.

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = 0$$

The solution of higher-order equations follows analogously. The nonhomogeneous equation

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = g(x)$$

can be solved by variation of parameters once the complimentary function, $y_c(x)$, is found.

THE SOLUTION

We try a solution of the form $y = x^m$, where m must be determined. The first and second derivatives are, respectively, $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Substituting into the differential equation, we find that x^m is a solution of the equation whenever m is a solution of the **auxiliary equation**

$$am(m-1) + bm + c = 0 \text{ or}$$

$$am^2 + (b-a)m + c = 0$$

THREE CASES

There are three cases to consider, depending on the roots of the auxiliary equation.

Case I: Distinct Real Roots

Case II: Repeated Real Roots

Case III: Conjugate Complex Roots

CASE I — DISTINCT REAL ROOTS

Let m_1 and m_2 denote the real roots of the auxiliary equation where $m_1 \neq m_2$. Then

$$y_1 = x^{m_1} \text{ and } y_2 = x^{m_2}$$

form a fundamental solution set, and the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

CASE II — REPEATED REAL ROOTS

Let $m_1 = m_2$ denote the real root of the auxiliary equation. Then we obtain one solution—namely,

$$y_1 = x^{m_1}$$

Using the formula from Section 4.2, we obtain the solution

$$y_2 = x^{m_1} \ln x$$

Then the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$$

CASE III — CONJUGATE COMPLEX ROOTS

Let $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$, then the solution is

$$y = C_1 x^{\alpha+\beta i} + C_2 x^{\alpha-\beta i}.$$

Using Euler's formula, we can obtain the fundamental set of solutions

$$y_1 = x^{\alpha} \cos(\beta \ln x) \text{ and } y_1 = x^{\alpha} \sin(\beta \ln x).$$

Then the general solution is

$$\begin{aligned} y &= c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x) \\ &= x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)] \end{aligned}$$

ALTERNATE METHOD OF SOLUTION

Any Cauchy-Euler equation can be reduced to an equation with constant coefficients by means of the substitution $x = e^t$.