Section 5.2

Damped Motion

DAMPED MOTION

The second-order differential equation

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \text{ or } \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where β is a positive damping constant, $2\lambda = \frac{\beta}{m}$, and $\omega^2 = \frac{k}{m}$ is the equation that describes **free damped motion**.

AUXILIARY EQUATION

The auxiliary equation is $m^2 + 2\lambda m + \omega^2 = 0$, and the corresponding roots are

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}, \qquad m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$

The solution to the differential equation is divided into three cases:

- **Case 1**: $\lambda^2 - \omega^2 > 0$
- **Case 2**: $\lambda^2 - \omega^2 = 0$
- **Case 3**: $\lambda^2 - \omega^2 < 0$

CASE 1 — OVERDAMPED MOTION

In this situation, the system is said to be **overdamped** since the damping coefficient β is large when compared to the spring constant k. The solution of the equation is

$$x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

CASE 2 — CRITICALLY DAMPED MOTION

The system is said to be **critically damped** since any slight decrease in the damping force would result in oscillatory motion. The solution of the equation is

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

CASE 3 — UNDERDAMPED MOTION

In this case the system is said the be **underdamped**, since the damping coefficient is small compared to the spring constant k. The roots are now complex

$$m_1 = -\lambda + \sqrt{\omega^2 - \lambda^2}\, i, \qquad m_2 = -\lambda - \sqrt{\omega^2 - \lambda^2}\, i$$

and so the solution to the equation is

$$x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2 - \lambda^2}\, t + c_2 \sin\sqrt{\omega^2 - \lambda^2}\, t\right)$$

ALTERNATIVE FORM FOR THE SOLUTION OF UNDERDAMPED MOTION

We can write any solution

$$x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2 - \lambda^2}\, t + c_2 \sin\sqrt{\omega^2 - \lambda^2}\, t\right)$$

in the alternative form

$$x(t) = Ae^{-\lambda t} \sin\left(\sqrt{\omega^2 - \lambda^2}\, t - \phi\right)$$

where $A = \sqrt{c_1^2 + c_2^2}$, and the phase angle ϕ is determined from the equations

$$\sin\phi = \frac{c_1}{A}, \cos\phi = \frac{c_2}{A}, \tan\phi = c_1/c_2$$

DAMPED AMPLITUDE, QUASI PERIOD, QUASI FREQUENCY

The **damped amplitude** is: $A = e^{-\lambda t}$

The **quasi period** is: $\frac{2\pi}{\sqrt{\omega^2-\lambda^2}}$

The **quasi frequency** is: $\frac{\sqrt{\omega^2-\lambda^2}}{2\pi}$